Costly auction entry, royalty payments, and the optimality of asymmetric designs*

Dan Bernhardt[†], Tingjun Liu[‡], and Takeharu Sogo[§]

March 2, 2020

Abstract

We analyze optimal auction mechanisms when bidders base costly entry decisions on their valuations, and bidders pay with a fixed royalty rate plus cash. With sufficient valuation uncertainty relative to entry costs, the optimal mechanism features asymmetry so that bidders enter with strictly positive but different (ex-ante) probabilities. When bidders are ex-ante identical, higher royalty rates—which tie payments more closely to bidder valuations—increase the optimal degree of asymmetry in auction design, further raising revenues. When bidders differ ex-ante in entry costs, the seller favors the low cost entrant; whereas when bidders have different valuation distributions, the seller favors the weaker bidder if entry costs are low, but not if they are high. Higher royalty rates cause the seller to favor the weaker bidder by less, and the strong bidder by more.

JEL-Classification:D44; G3
Keywords: Auctions with participation costs; Royalty payments; Optimal auctions; Asymmetric auctions; Heterogeneous bidders

*We are grateful to the anonymous referees and the associate editor for their constructive input. We thank Yu Awaya, Vivek Bhattacharya, Stephen Ching, Vijay Krishna, Jingfeng Lu, Hidenori Takahashi, and numerous seminar and conference participants for helpful comments. Liu gratefully acknowledges financial support from Hong Kong General Research Fund (project number 17502717). Sogo gratefully acknowledges financial support by JSPS KAKENHI Grant Number JP19K13659.

[†]University of Illinois, danber@illinois.edu and University of Warwick, M.D.Bernhardt@warwick.ac.uk
[‡]Faculty of Business and Economics, The University of Hong Kong, tjliu@hku.hk
[§]SKEMA Business School – Université Côte d'Azur, takeharu.sogo@skema.edu

1 Introduction

This paper investigates the optimal design of auctions when bidders base costly entry decisions on their valuations, and bidders pay with a fixed royalty rate plus cash. Such auctions are common: Skrzypacz (2013) reports oil and gas lease auctions typically feature equity payments in the form of royalties; Andrade, Mitchell, and Stafford (2001) report that 70% of mergers and acquisitions involve some equity; and, as Gorbenko and Malenko (2011) and Skrzypacz (2013) highlight, venture capital financing, procurement auctions, and lead-plaintiff auctions often use similar payment structures.

We establish that as long as there is sufficient valuation uncertainty relative to entry costs, the optimal mechanism features asymmetry so that bidders enter with strictly positive but different (ex-ante) probabilities. When bidders are ex-ante identical, higher royalty rates—which tie payments more closely to bidder types—increase the optimal degree of asymmetry in auction design, allowing the seller to raise revenues further. When bidders differ ex ante, strategic interactions become more subtle. We show that the optimal bidder to favor, the extent of such favoritism, and the impact of the royalty rate hinge on the nature of the heterogeneity and the size of entry costs. We provide a unified intuition for the driving forces underlying these findings.

In our setting, bidders have independent and private valuations for the asset, and know their valuations before incurring entry costs, as in Samuelson (1985) or Sogo, Bernhardt, and Liu (2016).[1] To highlight the basic insights and tradeoffs, we first examine a scenario with ex-ante identical bidders and symmetric equilibria. Paying for entry is costly and duplicative: the entry costs of all bidders save the winner are wasted. The optimal mechanism trades off between the increased rents that more entrants can bring versus the higher total entry costs incurred by more bidders. If valuation uncertainty for bidders is modest, so are the welfare gains from greater selection, but the probability that the asset goes unsold rises with the number of potential bidders. We provide sufficient conditions under which a seller should restrict entry to a single bidder, setting a take-it-or-leave-it price.[2]

[1]In the appendix we analyze auction designs when bidders only learn their valuations after *after* entering.

[2]This result is consistent with Gentry and Stroup (2019). Their estimates reveal that the relative performance of auctions over negotiations for corporate takeovers is higher when uncertainty about target

More typically, uncertainty over bidder valuations is more extensive relative to entry costs. In such a setting with two bidders, we identify mild conditions under which asymmetric auctions that favor one bidder over the other are optimal, so that bidders enter with strictly positive but different ex-ante probabilities. The intuition underlying the tradeoffs of introducing asymmetric entry thresholds is simple and, as we shall show, robust. Consider a small spread away from a symmetric mechanism with entry thresholds where each bidder enters with probability $p \in (0, 1)$ to an asymmetric mechanism in which one bidder enters with probability $p+\Delta p$ and the other enters with probability $p-\Delta p$. Introducing this asymmetry reduces the probability of no sale from $(1-p)^2$ to $(1-p)^2 - (\Delta p)^2$, while leaving total expected entry costs unchanged. However, it forsakes choice when the higher valuation bidder is excluded.

The optimal design balances these gains and costs. To encourage entry by a bidder with a low valuation, the seller sets a higher reserve for the other bidder. This preserves a high probability of trade, while obtaining efficient allocations when the handicapped bidder has a high valuation. Due to the rival's handicap, the bidder facing a low reserve is willing to enter even with a low valuation, understanding that the probability of competition is not so high.

Higher royalty rates increase the degree of asymmetry in the optimal design. This reflects that spreading entry thresholds raises total bidder profits at a seller's expense. However, higher royalty rates offset this by reducing bidder profit in the optimal design (reflecting the Demarzo, Kremer and Skrzypacz (hereafter DKS, 2005) logic that steeper securities enhance seller revenues). Higher royalty rates differentially reduce the profit of a bidder who faces a lower reserve when its valuation is high. Thus, they reduce the value attached by a seller to competition by another bidder, making it optimal to set a higher reserve for that bidder, increasing the asymmetry.[3] Higher royalty rates increase seller profit for a given degree of asymmetry, and when the degree is set optimally, seller revenues are further enhanced.

To reinforce how higher royalty rates increase the optimal degree of asymmetry we allow

values is extensive.

[3]The asymmetry refers to direct-mechanisms. In the analogous costless entry setting, Myerson (1981) shows the optimal direct-mechanism is necessarily symmetric when bidders are ex-ante identical. Deb and Pai (2017) show in a no entry cost setting that if one maintains interim individual rationality, but relaxes ex-post individual rationality so that a winning bidder can pay more than the asset's expected value, then an asymmetric direct-mechanism can almost always be implemented via a symmetric indirect-mechanism in which payments hinge on the bids of all bidders.

for a quadratic cost $\gamma(\hat{\theta}_2 - \hat{\theta}_1)^2$ of adopting asymmetric mechanisms with $\hat{\theta}_1 \neq \hat{\theta}_2$. Asymmetric mechanisms remain optimal if and only if $\gamma < \gamma^*(\alpha)$, where $\gamma^*(\alpha)$ strictly increases with the royalty rate α: when royalty rates are higher, asymmetric mechanism remain optimal even when the costs of implementing asymmetric mechanisms are higher.

In practice, bidders typically differ ex-ante from a seller's perspective. In such settings, the questions become: which bidder does a seller want to favor, and to what extent? We analyze how different forms of bidder heterogeneity affect the optimal design in a two-bidder, uniformly-distributed valuation setting. We first show that when bidders have different entry costs, the seller favors the low cost bidder, because a seller indirectly pays the entry costs.

In contrast, when one bidder has a higher upper support for valuations than the other, strategic considerations become subtle. We show that if entry costs are low, a seller favors the weaker bidder, but once entry costs are sufficiently high, the seller favors the stronger bidder.

Two considerations enter a seller's decision-making: rent-extraction concerns and efficiency concerns. Optimal entry thresholds reflect virtual valuations, not actual valuations. When entry costs are low, then reflecting the (entry-cost free) logic of Myerson (1981), a seller favors the weaker bidder to reduce the stronger bidder's ability to extract informational rents when its valuation is high. The seller also wants to reduce the probability of an inefficient, no sale outcome. Concretely, suppose the probability of entry for bidder i changes from p_i to $p_i + \Delta p$, while that for bidder j changes from p_j to $p_j - \Delta p$. Then, the probability of no sale becomes

$$[1 - (p_i + \Delta p)][1 - (p_j - \Delta p)] = (1 - p_i)(1 - p_j) - (\Delta p)^2 - (p_i - p_j)\Delta p.$$

Thus, when $p_i > p_j$, increasing p_i and decreasing p_j is more effective at reducing the probability of no sale than the reverse. When entry costs are low, the weaker bidder is more likely to enter, so the rent-extraction logic underlying why the weaker bidder is favored is initially *reinforced* by efficiency considerations, magnifying the degree to which the weaker bidder is favored—beyond the degree prescribed by Myerson when entry is costless. However, as entry costs rise, so do the optimal cutoffs, and eventually the stronger bidder becomes more likely to enter. Hence, once entry costs are high enough, it becomes optimal to favor the stronger bidder. Thus, there is a non-monotone relationship between entry costs and the degree to which the weaker bidder is favored (or disfavored).

Greater royalty rates reduce rent-extraction concerns because they lower bidder information rents. As a result, when entry costs are low, greater royalty rates make it optimal to favor the weaker bidder by less, reducing the optimal degree of asymmetry; and when entry costs are high, greater royalty rates make it optimal to favor the stronger bidder by more, raising the optimal degree of asymmetry.

Our analysis provides theoretical foundations for negotiated break-up fees that favor one bidder over another in a takeover auction or the favoring of a particular supplier in procurement auctions. A firm seeking a buyer often elicits an initial bid from one bidder by promising to reimburse it for its efforts if it is outbid, thereby committing the target firm to excluding other bidders unless their valuations are sufficiently high.[4] In procurements the US government often provides preferential treatment to domestic firms and small businesses. Ayres and Cramton (1996) show such preferential treatments enhance seller revenues in auctions for paging licenses by the FCC, in which winning bids of favored bidders are subsidized by a fixed rate.[5] In a setting with ex-ante asymmetric bidders, McAfee and McMillan (1989) show that favoring ex-ante weaker bidders can enhance auction revenues. In addition to revealing the revenue-enhancing effect of bid preference policies in a general framework with heterogeneous bidders, our endogenous entry model provides guidance on how the optimal design regarding the identity of which bidder to favor and the extent of that favoritism should vary with the sizes of entry costs and royalty rates, and the nature of bidder asymmetry.

We contribute to research on the optimal design of security auctions without entry costs, research on standard (symmetric) security auction designs with entry costs, and research on cash auctions with entry costs. Absent entry costs, Cremer (1987) shows that optimal

[4]Bates and Lemmon (2003) find that 37% of the firms in their sample sign break-up/termination fees and 17% grant lock-up options. Note that our analysis presumes a fixed royalty rate, whereas in takeovers the royalty rate may be endogenously chosen by a bidder. When bidders are cash constrained and can only offer equity, bidders with larger investment costs (standalone values) effectively correspond to lower royalty rates in our model, resulting in heterogeneities among bidders when bidders have different investment costs (see footnote 15 for further discussions).

[5]For empirical analyses of the value of bid preference policies in settings with unknown valuations and costly entry, see Krasnokutskaya and Seim (2011), Athey, Coey, and Levin (2013), and Nakabayashi (2013).

securities auctions extract almost all surplus; and DKS show that if a seller restricts bids to an ordered set of securities and uses a standard auction format, then steeper securities yield higher revenues. Liu (2016) identifies the optimal mechanism when bidders are heterogeneous and pay with equities, generalizing Myerson (1981) from cash auctions to equity auctions. Skrzypacz (2013) reviews the security-bid auction literature. Our contribution is to identify optimal mechanisms when bidders bid with cash plus royalty payments and incur entry costs, so that entry is endogenous.[6]

Fishman (1988) studies takeover contests in which an acquirer faces a potential rival that must incur a cost to learn its target firm valuation, showing that a high-valuation acquirer may offer the target a high price to preemptively discourage a rival from becoming informed. Marquez and Singh (2013) investigate club bidding in private equities, and how entry costs affect club formation and seller profits. Gorbenko and Malenko (2011) endogenize competition between sellers in the design of security-bid auctions when bidders learn valuations after incurring entry costs. Sogo, Bernhardt, and Liu (2016) examine entry decisions in security-bid auctions when bidders know their valuations prior to entering. These papers study standard auction formats with entry costs. In contrast, we analyze optimal cash-plus-royalty auction mechanisms, optimizing over the entire space of symmetric and asymmetric mechanisms.

For cash auctions, Samuelson (1985) was the first to note that a seller can gain by restricting the potential number of bidders. Other cash auction papers that analyze the benefit of regulating entry include Ye (2007); Bhattacharya, Roberts, and Sweeting (2014); Sweeting and Bhattacharya (2015); and Quint and Hendricks (2018). Stegeman (1996) studies ex-ante efficient cash mechanisms with entry costs. Lu (2009) considers revenue-maximizing cash mechanisms with ex-ante identical bidders, showing that it suffices to focus on the class of threshold-entry mechanisms. Lu (2009) and Celik and Yilankaya (2009) provide examples of optimal mechanisms in which bidders enter with asymmetric and positive probabilities. In contrast, we examine the optimal design of auctions featuring cash and royalty payments, both when bidders are ex-ante identical and when they differ. We uncover how the nature

[6]Our insights concerning how α affects the direction and extent of the optimal asymmetry should extend to auctions where bids are from ordered sets of securities (as in DKS) in terms of how the steepness of the securities affects the optimal design.

of ex-ante bidder heterogeneity, and the sizes of entry costs and royalty rates interact to jointly determine the optimal design regarding which bidder to favor and the extent, and we provide a unified intuition for these results.

2 Model

There is a risk-neutral seller and $n \geq 1$ risk-neutral potential bidders. The indivisible asset being auctioned has a normalized value of zero if retained by the seller. Bidder i incurs cost $\phi_i > 0$ from entering the auction. If bidder i acquires the asset, then it will yield a stochastic payoff of Z_i at date 2. Bidders pay using combinations of cash and royalty/equity where the royalty payment by winner i is αZ_i, i.e., the royalty rate $\alpha \in [0, 1)$ is the same for all bidders.

At date 0, each potential bidder i receives a private signal $\theta_i = E(Z_i|\theta_i)$ about the expected asset payoff if i wins it. Signals are independently distributed, with θ_i distributed according to cdf $F_i(\theta_i)$ and pdf $f_i(\theta_i)$, where $f_i(\theta_i) > 0$ is differentiable for $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$. For simplicity, we assume a regularity condition that any bidder i's virtual valuation in pure cash auctions (Myerson 1981), $\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$, strictly increases in θ_i, and that $\underline{\theta}_i - \frac{1}{f_i(\underline{\theta}_i)} > \phi_i$ (i.e., virtual valuations exceed entry costs). At date 1, after receiving signals, potential bidders simultaneously decide whether to enter the auction.

In cash auctions with costly entry, Stegeman (1996, Lemma 1) establishes that it is without loss of generality to restrict attention to *semidirect* mechanisms in which messages consist of types augmented by null messages; and Lu (2009, Lemma 1) establishes that among all semidirect mechanisms, the seller's expected revenue is maximized by cutoff entry rules in which only types $\theta_i \geq \hat{\theta}_i$ of each bidder i participate for some $\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$. This result that it is optimal to have higher types participate extends directly to our setting where the winning bidder makes both cash and royalty payments, because the royalty rate is fixed. Thus, without loss of generality, we restrict attention to semidirect mechanisms $(\mathbf{W}, \mathbf{T}; \alpha)$ that induce a truthful equilibrium with entry cutoffs $(\hat{\theta}_1, \cdots, \hat{\theta}_n)$.

Concretely, after observing θ_i, bidder i sends a message $m_i \in M_i \equiv [\underline{\theta}_i, \overline{\theta}_i] \cup \emptyset$, where $\emptyset$ is a null message (i.e., no entry). Let $M \equiv \times_i M_i$. The mechanism $(\mathbf{W}, \mathbf{T}; \alpha)$ is comprised of (i) a profile of winning rules $W = (W_1, \ldots, W_n)$, where $W_i : M \to [0, 1]$ is the probability that bidder i wins given $m = (m_1, .., m_i, ..., m_n)$ and $\sum_i W_i(m) \leq 1$, $\forall m \in M$; (ii) a profile of monetary (cash) transfer rules $T = (T_1, \ldots, T_n)$ with $T_i : M \to \mathbb{R}$; and (iii) a royalty rate α that determines the payment αZ_i when bidder i wins the auction and the asset pays Z_i. Thus, when $m = (m_1, \ldots, m_n) \in M$ is the profile of reported messages, if $m_i \in [\underline{\theta}_i, \overline{\theta}_i]$, bidder i incurs entry cost ϕ_i, pays cash $T_i(m)$ to the seller, and wins the auction with probability $W_i(m)$, in which case he makes royalty payment αZ_i to the seller; and if $m_i = \emptyset$, bidder i does not enter, receives zero payoff, and $W_i(m) = T_i(m) = 0$. This formulation corresponds to a setting where a seller may ask losing bidders to make monetary transfers, but cannot assign the asset or make monetary transfers to non-entrants.

Truthful bidding yields that $m_i(\theta_i) = \theta_i$ if $\theta_i \in [\hat{\theta}_i, \overline{\theta}_i]$ and $m_i(\theta_i) = \emptyset$ if $\theta_i \in [\underline{\theta}_i, \hat{\theta}_i)$. We define: $m(\theta) \equiv \times_i m_i(\theta_i)$, $\theta_{-i} \equiv (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \in \times_{j \neq i}[\underline{\theta}_j, \overline{\theta}_j]$, $m_{-i}(\theta_{-i}) \equiv \times_{j \neq i} m_j(\theta_j)$, $f_{-i}(\theta_{-i}) \equiv \Pi_{j \neq i} f_j(\theta_j)$, and $f(\theta) \equiv \Pi_i f_i(\theta_i)$. Consider bidder i's expected profit π_i when all other bidders follow their equilibrium strategies. If he reports $\emptyset$ (i.e., if i does not participate), then $\pi_i(\emptyset|\theta_i) = 0$. If bidder i instead reports $\theta_i' \in [\underline{\theta}_i, \overline{\theta}_i]$, then

$$\pi_i(\theta_i'|\theta_i) \equiv G_i(\theta_i')(1-\alpha)\theta_i - \int_{\theta_{-i}} T_i(\theta_i', m_{-i}(\theta_{-i})) f_{-i}(\theta_{-i})\, d\theta_{-i} - \phi_i, \tag{1}$$

where

$$G_i(\theta_i') \equiv \int_{\theta_{-i}} W_i(\theta_i', m_{-i}(\theta_{-i})) f_{-i}(\theta_{-i})\, d\theta_{-i} \tag{2}$$

is i's expected winning probability if he reports θ_i' and other bidders follow their equilibrium strategies.

Individual rationality for type $\theta_i < \hat{\theta}_i$ is trivially satisfied because non-participants receive zero profit. Individual rationality for type $\theta_i \geq \hat{\theta}_i$ requires

$$\pi_i(\theta_i) \equiv \pi_i(\theta_i|\theta_i) \geq 0, \ \forall \theta_i \geq \hat{\theta}_i, \tag{3}$$

where $\pi_i(\theta_i)$ denotes i's equilibrium profit. Incentive compatibility for type $\theta_i \geq \hat{\theta}_i$ requires

$$\pi_i(\theta_i) = \max_{\theta' \in [\underline{\theta}_i, \overline{\theta}_i]} \pi_i(\theta_i'|\theta_i), \ \forall \theta_i \geq \hat{\theta}_i; \tag{4}$$

Incentive compatibility for type $\theta_i < \hat{\theta}_i$ requires

$$\max_{\theta' \in [\underline{\theta}_i, \overline{\theta}_i]} \pi_i(\theta_i'|\theta_i) \leq 0, \ \forall \theta_i < \hat{\theta}_i. \tag{5}$$

A semidirect mechanism $(\mathbf{W}, \mathbf{T}; \alpha)$ that induces entry cutoffs $(\hat{\theta}_1, \cdots, \hat{\theta}_n)$ is *feasible* if it satisfies the individual rationality and incentive compatibility constraints (3), (4), and (5). The seller's problem is to find a feasible semidirect mechanism with associated entry cutoffs that maximizes her expected revenue

$$\Pi_s = \int_\theta \sum_i [T_i(m(\theta)) + W_i(m(\theta)) \cdot \alpha\theta_i] f(\theta)\, d\theta. \tag{6}$$

Characterization. By (1), we have that for all $\theta_i', \theta_i \geq \hat{\theta}_i$,

$$\pi_i(\theta_i'|\theta_i) = \pi_i(\theta_i') + (1-\alpha)G_i(\theta_i')(\theta_i - \theta_i').$$

Plugging this into the incentive compatibility constraint (4), yields

$$\pi_i(\theta_i) \geq \pi_i(\theta_i') + (1-\alpha)G_i(\theta_i')(\theta_i - \theta_i'), \ \forall \theta_i', \theta_i \geq \hat{\theta}_i. \tag{7}$$

Equation (7) shows that a high-type bidder earns more rents than a low-type bidder by at least $(1-\alpha)G_i(\theta_i')(\theta_i - \theta_i')$, where $\theta_i > \theta_i' \geq \hat{\theta}_i$. Here, the factor $(1-\alpha)$ reflects that the royalty rate ties the payment value to the bidder's valuation, reducing the bidder's rents. Using standard approaches (see Myerson 1981; Krishna 2010), we now establish:

Lemma 1 *In any feasible semidirect mechanism that induces entry cutoffs* $(\hat{\theta}_1, \cdots, \hat{\theta}_n)$,

1. *The expected profit of bidder i of type θ_i is*

$$\pi_i(\theta_i) = \pi_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} (1-\alpha)G_i(t)dt, \ \forall \theta_i \geq \hat{\theta}_i, \ \forall i. \tag{8}$$

2. *The seller's expected revenue is*

$$\Pi_s = \int_\theta \left(\sum_i \xi_i(\theta_i; \alpha) W_i(m(\theta)) \right) f(\theta)\, d\theta - \sum_i \left(\phi_i + \pi_i(\hat{\theta}_i) \right) (1 - F_i(\hat{\theta}_i)), \tag{9}$$

where

$$\xi_i(\theta_i;\alpha) \equiv \theta_i - (1-\alpha)\,\frac{1-F_i(\theta_i)}{f_i(\theta_i)}. \tag{10}$$

All proofs are contained in the Appendix. This lemma extends the standard Myerson result to auctions with royalty payments and entry costs. Equation (10) is the virtual valuation when bidders pay with a combination of cash and a fixed royalty rate α (Myerson 1981 and Liu 2016), where the factor $(1-\alpha)$ reflects that security payments allow the seller to extract more rents from bidders than pure cash payments, reducing bidder profit. The factor $1 - F_i(\hat{\theta}_i)$ in the second term of (9) is the (ex-ante) probability that bidder i participates. Bidder i's expected entry cost of $(1-F_i(\hat{\theta}_i))\phi_i$ represents a loss of social surplus that is eventually borne by the seller via the bidder's individual rationality requirement. That is, if the seller wants bidder i to participate, it must design the mechanism so that bidder i expects to gain at least ϕ_i upon participation, which results in a reduction in expected payments. In the second-price auction used to implement the optimal asymmetric mechanism, the seller addresses bidder individual rationality by directly reimbursing bidders to cover their entry costs.

Lemma 2 *In the optimal mechanism among all feasible semidirect mechanisms that induce entry cutoffs* $(\hat{\theta}_1, \cdots, \hat{\theta}_n)$*:*

1. *Among bidders with* $\theta_j \geq \hat{\theta}_j$*, the bidder* i *with the highest* $\xi_i(\theta_i;\alpha)$ *receives the asset.*

2. $\pi_j(\hat{\theta}_j) = 0$ *for all* j*.*

3. *The seller's expected revenue is*

$$\Pi_s = \int_\theta \max\left\{\mathbf{1}_{\{\theta_1 \geq \hat{\theta}_1\}}\xi_1(\theta_1;\alpha), ..., \mathbf{1}_{\{\theta_n \geq \hat{\theta}_n\}}\xi_n(\theta_n;\alpha)\right\} f(\theta)\, d\theta - \phi_i \sum_i (1-F_i(\hat{\theta}_i)), \tag{11}$$

where $\mathbf{1}_{\{\theta_i \geq \hat{\theta}_i\}}$ *is an indicator function.*

Thus, in the optimal mechanism, the seller's revenue is the expected value of the highest virtual valuation among entrants net of total entry costs. The seller's problem reduces to the identification of entry cutoffs $(\hat{\theta}_1, \cdots, \hat{\theta}_n)$ that maximize (11).

3 Ex-ante Identical Bidders

In this section we consider ex-ante identical bidders, so that $\phi_i = \phi$ and $f_i = f$ for all i.

3.1 Optimal Symmetric Mechanism

To begin, we consider symmetric mechanisms that induce the same entry cutoff $\hat{\theta}_i = \hat{\theta}$ for all bidders. The distribution of the highest type θ_n^1 among n potential bidders is $F^n(\theta_n^1)$. From (11), the expected revenue in the optimal mechanism with cutoff $\hat{\theta}$ is

$$\Pi^*(\hat{\theta}) \equiv \int_{\hat{\theta}}^{\bar{\theta}} \xi(\theta;\alpha) dF^n(\theta) - n\phi(1 - F(\hat{\theta})). \tag{12}$$

The entry cutoff $\hat{\theta}^{opt}$ that maximizes expected seller payoffs $\Pi^*(\hat{\theta})$ is characterized by:

$$\begin{cases} \xi(\hat{\theta}^{opt};\alpha)F^{n-1}(\hat{\theta}^{opt}) = \phi,\ \hat{\theta}^{opt} \in (\underline{\theta}, \bar{\theta}) & \text{if} \quad n \geq 2 \\ \hat{\theta}^{opt} = \underline{\theta} & \text{if} \quad n = 1. \end{cases} \tag{13}$$

To see this, observe that $\frac{d\Pi^*(\hat{\theta})}{d\hat{\theta}} = -nF^{n-1}(\hat{\theta})f(\hat{\theta})\eta(\hat{\theta};\alpha,\phi)$ by (12), where $\eta(\hat{\theta};\alpha,\phi) \equiv \xi(\hat{\theta};\alpha) - \frac{\phi}{F^{n-1}(\hat{\theta})}$. Because $\xi(\hat{\theta};\alpha)$ increases in $\hat{\theta}$, so does $\eta(\hat{\theta};\alpha,\phi)$, and $\eta(\bar{\theta};\alpha,\phi) > 0$. For $n \geq 2$, $\lim_{\hat{\theta}\to\underline{\theta}} \eta(\hat{\theta};\alpha,\phi) < 0$. Therefore, there is a unique $\hat{\theta}^{opt} \in (\underline{\theta}, \bar{\theta})$ such that $\eta(\hat{\theta}^{opt};\alpha,\phi) = 0$. Inspection reveals that $\frac{d\Pi^*(\hat{\theta})}{d\hat{\theta}} > 0$ if $\hat{\theta} < \hat{\theta}^{opt}$, and $\frac{d\Pi^*(\hat{\theta})}{d\hat{\theta}} < 0$ if $\hat{\theta} > \hat{\theta}^{opt}$. Thus, $\hat{\theta}^{opt}$ maximizes Π^*. For $n = 1$, $F^{n-1}(\hat{\theta}) = 1$ and $\frac{d\Pi^*(\hat{\theta})}{d\hat{\theta}} < 0$, implying that $\underline{\theta}$ maximizes Π^*.

Increasing α reduces both the informational rents of bidders and the difference between virtual and real valuations (both ξ and η increase). It follows that greater royalty rates reduce the optimal entry cutoff. To highlight this starkly, consider the limit case of $\alpha \to 1$ when $n \geq 2$. Then $\xi(\theta;\alpha = 1) = \theta$ and (13) reduces to $\hat{\theta}^{opt}F^{n-1}(\hat{\theta}^{opt}) = \phi$. As the entry cost ϕ decreases, so does the optimal entry cutoff $\hat{\theta}^{opt}$, going to $\hat{\theta}^{opt}_{\phi=0} = \underline{\theta}$ as $\phi \to 0$. Then (12) yields

$$\Pi^*(\hat{\theta}^{opt}_{\phi=0;\alpha\to 1}) = \int_{\underline{\theta}}^{\bar{\theta}} \theta dF^n(\theta).$$

Intuitively, when entry is costless, it is optimal to always award the asset (i.e., $\hat{\theta}^{opt}_{\phi=0} = \underline{\theta} > 0$); and as $\alpha \to 1$, the optimal mechanism leaves no rents to bidders, so seller revenue equals the social welfare gain created by the asset's allocation. By (12) again, comparing this costless-entry benchmark and our costly-entry setting where $\phi > 0$ yields

$$\Pi^*(\hat{\theta}^{opt}_{\phi=0;\alpha\to1}) - \Pi^*(\hat{\theta}^{opt}_{\phi>0;\alpha\to1}) = n\phi(1 - F(\hat{\theta}^{opt})) + \int_{\underline{\theta}}^{\hat{\theta}^{opt}} \theta dF^n(\theta).$$

Costly entry reduces a seller's maximum rents in two ways. The first term captures the direct cost of entry. The second term reflects an indirect efficiency loss: as reasoned above, costly entry impairs the efficiency of allocations as it is no longer optimal to always award the asset—the threshold $\hat{\theta}^{opt}$ exceeds $\underline{\theta}$, so a seller foregoes some socially optimal trades.

Restricting entry. We show that with limited uncertainty about bidder valuations, conserving on entry costs by restricting entry to one bidder may be optimal; but with more extensive uncertainty, asymmetric mechanisms that handicap some bidders are best.

To start, we motivate the tradeoffs associated with exclusion faced by a seller. Fixing the sum of bidders' entry probabilities, and thus the total expected direct costs of entry (the second term in equation (12)), but increasing the number of potential bidders n, has two opposing effects on social welfare. Conditional on some bidder entering, welfare gains typically rise with n, as the expected winner's valuation is higher due to a *greater-selection effect*. However, the probability that no bidder enters and no trade occurs also rises with n, which harms welfare. Intuition for this *deterred-entry effect* derives from a simple inequality:

Lemma 3 *Given $n > 1$ numbers $p_1, ..., p_n \in (0, 1)$, $1 - \min\{\sum_{j=1}^n p_j, 1\} < \Pi_{j=1}^n (1 - p_j)$.*

Corollary 1 *Suppose that rather than $n > 1$ potential bidders entering with (ex-ante) probabilities $p_1, ..., p_n \in (0, 1)$, a single potential bidder enters with probability $min\{\sum_{j=1}^n p_j, 1\}$. Then, the asset is strictly more likely to be sold, and total expected entry costs are weakly less.*

Because expected total entry costs equal the sum of participation probabilities times ϕ, a single potential bidder can achieve a higher probability of sale while paying an entry cost that does not exceed the total entry costs of n potential bidders (i.e., $\min\{\Sigma_{j=1}^n p_j, 1\} \leq \Sigma_{j=1}^n p_j$). This is the deterred-entry effect of having more potential bidders. The greater-selection effect—the value of sampling more bidders to improve the draw of entrants with high valuations—increases with the extent of variation in bidder valuations. When the variation in valuations is small enough relative to the entry cost ϕ, the deterred-entry effect dominates the greater-selection effect in an extreme form:

Proposition 1 *Let bidder valuations be distributed on* $[\bar{\theta}-\epsilon,\bar{\theta}]$*. If* $\epsilon<\phi\left(1-\frac{\phi}{\bar{\theta}}\right)$*, then for any* α*, expected seller profits are strictly higher in the optimal mechanism with one potential bidder than in any symmetric mechanism with* $n\geq 2$ *potential bidders.*

The proposition provides a sufficient condition that holds for all α and only depends on ϵ but not the other details of $F(\cdot)$. To illustrate the result, revisit the case of $\alpha\to 1$ and take the limit $\epsilon\to 0$. If there are $n\geq 2$ potential bidders, then by $\xi(\theta;\alpha=1)=\theta$, (13) reduces to $\hat{\theta}^{opt}F^{n-1}(\hat{\theta}^{opt})=\phi$, yielding

$$F^{n-1}(\hat{\theta}^{opt})=\frac{\phi}{\hat{\theta}^{opt}}.$$

Taking limits on both sides yields

$$\lim_{\epsilon\to 0}F^{n-1}(\hat{\theta}^{opt})=\lim_{\epsilon\to 0}\frac{\phi}{\hat{\theta}^{opt}}=\frac{\phi}{\bar{\theta}},$$

where the last equality holds because $\hat{\theta}^{opt}$ approaches $\bar{\theta}$ as $\epsilon\to 0$. Thus,

$$\lim_{\epsilon\to 0}F\left(\hat{\theta}^{opt}(n)\right)=\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}},$$

where we index optimal entry cutoff $\hat{\theta}^{opt}$ by n (for $n\geq 2$). Indexing profits Π by n, (12) yields

$$\begin{aligned}
\lim_{\epsilon\to 0}\Pi\big(\hat{\theta}^{opt}(n),n\big) &= \lim_{\epsilon\to 0}\bar{\theta}\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}dF^{n}(\theta)-\lim_{\epsilon\to 0}n\phi(1-F(\hat{\theta}^{opt}(n)))\\
&=\lim_{\epsilon\to 0}\bar{\theta}\left(1-F^{n}\left(\hat{\theta}^{opt}(n)\right)\right)-\lim_{\epsilon\to 0}n\phi(1-F(\hat{\theta}^{opt}(n)))\\
&=\bar{\theta}\left(1-\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{n}{n-1}}\right)-n\phi\left(1-\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\right)\\
&=\bar{\theta}-\bar{\theta}\frac{\phi}{\bar{\theta}}\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}-\phi\left(1-\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\right)-(n-1)\,\phi\left(1-\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\right)\\
&=\bar{\theta}-\phi-(n-1)\phi\bigg(1-\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\bigg),
\end{aligned}\tag{14}$$

where the first equality follows from $0\leq\bar{\theta}-\theta\leq\epsilon$ for all $\theta\in[\hat{\theta}^{opt},\bar{\theta}]$ and $\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}dF^{n}(\theta)\leq 1$.[7]

[7]More concretely, note that $0\leq\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}\bar{\theta}dF^{n}(\theta)-\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}\theta dF^{n}(\theta)\leq\epsilon\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}dF^{n}(\theta)\leq\epsilon$. Hence, as ϵ approaches zero, $\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}\bar{\theta}dF^{n}(\theta)-\int_{\hat{\theta}^{opt}(n)}^{\bar{\theta}}\theta dF^{n}(\theta)$ goes to zero, establishing the first equality of (14).

For $n = 1$, because $\hat{\theta}^{opt}(1) = \underline{\theta} = \bar{\theta} - \varepsilon$ by (13), (12) yields

$$\lim_{\epsilon \to 0} \Pi\big(\hat{\theta}^{opt}(1), 1\big) = \bar{\theta} \int_{\bar{\theta}-\epsilon}^{\bar{\theta}} dF(\theta) - \phi = \bar{\theta} - \phi, \tag{15}$$

where the first equality holds by a similar argument as that for the first equality of (14). By (14) and (15), expected revenue strictly decreases in n, maximizing at $n = 1$.[8]

3.2 Optimality of Asymmetric Mechanisms

We now show that when the possible valuations of potential bidders differ by enough (vis à vis entry costs), then while excluding bidders is not optimal, neither is a symmetric mechanism. For simplicity, we focus on $n = 2$. We first consider $\alpha \to 1$, where the tradeoffs are simpler. As $\alpha \to 1$, seller revenue in (11) in the optimal (possibly asymmetric) mechanism reduces to

$$\begin{aligned} &\int_{\hat{\theta}_2}^{\bar{\theta}} \int_{\hat{\theta}_1}^{\bar{\theta}} \max\{\theta_1, \theta_2\}\, dF(\theta_1)\, dF(\theta_2) + F(\hat{\theta}_2) \int_{\hat{\theta}_1}^{\bar{\theta}} \theta_1 dF(\theta_1) + F(\hat{\theta}_1) \int_{\hat{\theta}_2}^{\bar{\theta}} \theta_2 dF(\theta_2) \\ &\quad - \phi(2 - F(\hat{\theta}_1) - F(\hat{\theta}_2)). \end{aligned} \tag{16}$$

The first term in (16) is the social welfare gain when both bidders enter, and the second and third are the social welfare gains when only one of the bidders enters.

We next establish a benchmark result for the case of $\alpha \to 1$ that asymmetric mechanisms that handicap one bidder are optimal when valuations can differ sufficiently:

Proposition 2 *Suppose (i) $\frac{df(\theta)}{f(\theta)} > -\frac{1}{(\bar{\theta}-\underline{\theta})} \ln \frac{\bar{\theta}}{(\bar{\theta}-\underline{\theta})}$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, i.e., the pdf does not decrease too quickly, and (ii) there is enough valuation uncertainty that $\bar{\theta} - E[\theta] > \phi$. Then, as α approaches 1, optimal mechanisms are asymmetric, with both bidders entering with strictly positive but different (ex-ante) probabilities. That is, $\hat{\theta}_1^{opt} \neq \hat{\theta}_2^{opt}$ and $\max\{\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt}\} < \bar{\theta}$.*

Proposition 2 provides conditions under which the optimal mechanism with ex-ante identical bidders is asymmetric, so that bidders enter with strictly positive but different (ex-ante) probabilities, a result that does not hinge on large entry costs (see condition (ii)). To see the logic, consider a small spread away from a symmetric mechanism with entry thresholds

[8] For $n \geq 2$, $m(1 - a^{1/m})$ increases in $m = n - 1$ for $a < 1$: letting $b = 1/m$, the derivative with respect to m is $1 - a^b(1 - \ln(a^b)) > 0$, as $\frac{d}{dx} x(1 - \ln(x)) = -\ln(x) > 0$ for $x < 1$, and $\lim_{x \to 1} x(1 - \ln(x)) \to 1$.

$\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}$, where each bidder enters with probability $p \in (0, 1)$, to an asymmetric mechanism with entry thresholds $\hat{\theta}_1 = \hat{\theta} - \epsilon$ and $\hat{\theta}_2 = \hat{\theta} + \epsilon^*$, where ϵ and ϵ^* are chosen so that one bidder enters with probability $p + \Delta p$ and the other enters with probability $p - \Delta p$.

Introducing this asymmetry reduces the probability of no sale from $(1-p)^2$ to $(1-p)^2 - (\Delta p)^2$, while leaving total expected entry costs unchanged.[9] However, it forsakes some choice when the higher valuation bidder is excluded. Condition (i) ensures that the density does not decline so quickly that the potential (and hence expected) value of that foregone choice is too high, making it always optimal to spread the cutoffs of a symmetric mechanism. Condition (ii) ensures that always excluding a bidder (in which case, it is optimal for the other bidder always to enter, because $\phi < \underline{\theta}$) is not optimal: the left-hand side of $\bar{\theta} - E[\theta] > \phi$ is the expected benefit of entry by a bidder with valuation $\bar{\theta}$ when the other bidder always enters, while the right-hand side is the cost.

The optimality of asymmetric mechanisms extends beyond optimal securities auctions that use steep securities ($\alpha \to 1$) to extract full rents, to hold for optimal mechanisms featuring any fixed royalty rate $\alpha \in [0, 1)$ plus cash. To highlight this, the remainder of the paper specializes to two bidders and sufficient uniform uncertainty over valuations on $[\underline{\theta}, \bar{\theta}]$:

Assumption A1: There are two potential bidders whose valuations are independently and uniformly distributed on $[\underline{\theta}, \bar{\theta}]$ with $2\underline{\theta} - \bar{\theta} > \phi$ and $\frac{\bar{\theta} - \underline{\theta}}{2} > \phi$.

In this uniform distribution setting, the condition $2\underline{\theta} - \bar{\theta} > \phi$ is equivalent to our earlier assumption that $\phi < \underline{\theta} - \frac{1}{f(\underline{\theta})}$. It ensures that with a single potential entrant, the optimal cutoff is $\underline{\theta}$.[10] The condition that $\frac{\bar{\theta} - \underline{\theta}}{2} > \phi$ corresponds to condition (ii) in Proposition 2 that there be sufficient valuation uncertainty, and condition (i) in Proposition 2 is trivially satisified in the uniform setting. We now show that the qualitative content of Proposition 2 holds for all fixed royalty rates $\alpha \in [0, 1]$ given only slightly stronger sufficient conditions:

Proposition 3 *Under **A1**, for any α,*

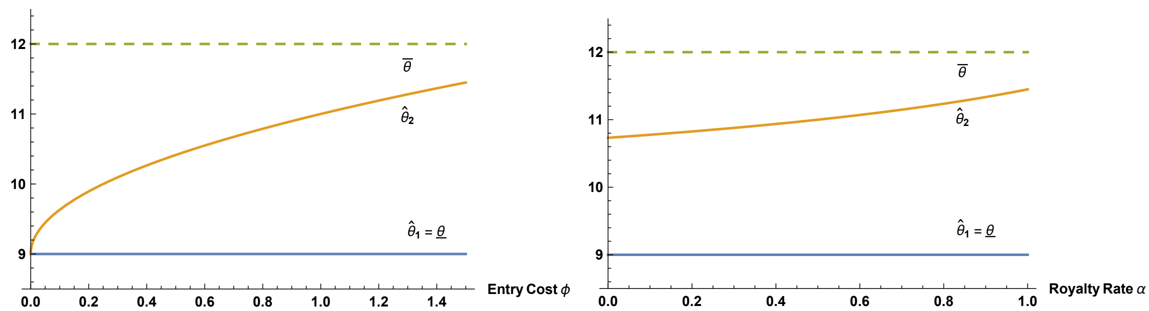
(*i*) *The optimal mechanism is asymmetric, with both bidders entering with strictly positive*

[9] This intuition leads to the optimality of asymmetric mechanism only when $\phi > 0$. Absent entry costs, the logic breaks down: the optimal mechanism features $p = 1$, so one cannot have a probability of $p + \Delta p$.

[10] Given this condition, the virtual valuation for a fixed royalty rate α plus cash, (10), calculated at $\underline{\theta}$, exceeds ϕ for any α.

but different probabilities.

(ii) If, in addition, $\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$*, then in the optimal mechanism, one bidder always enters and the other bidder enters with an ex-ante probability strictly between zero and one, where the cutoff valuation is* $\underline{\theta} + \sqrt{\frac{2\phi(\bar{\theta}-\underline{\theta})}{2-\alpha}} \in (\underline{\theta}, \bar{\theta})$*. This cutoff strictly increases in* α*.*



(a) $\hat{\theta}_1$ and $\hat{\theta}_2$ as a function of ϕ ($\alpha = 0.5$). (b) $\hat{\theta}_1$ and $\hat{\theta}_2$ as a function of α ($\phi = 1$).

Figure 1: Optimal entry cutoffs $\hat{\theta}_1$ and $\hat{\theta}_2$ as a function of ϕ and α when bidder valuations are independently uniformly distributed on $[9, 12]$ and $\phi < 1.5$ so **A1** holds.

Figure 1 illustrates Proposition 3 that one bidder always enters ($\hat{\theta}_1 = \underline{\theta}$) and the other bidder enters only when its valuation exceeds a cutoff $\hat{\theta}_2$. It shows how the optimal cutoff $\hat{\theta}_2$ increases in both ϕ and α; that is, greater ϕ or α increases the optimal degree of asymmetry.

When $\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$, a slight increase in the spread between $\hat{\theta}_1$ and $\hat{\theta}_2$ raises seller revenues—the benefit of increased probability of including one bidder swamps the cost of increased probability of excluding the other bidder when its valuation is higher. But then the optimum features a boundary solution. However, as explained below Proposition 2, the condition $\bar{\theta} - E[\theta] = \frac{1}{2}\left(\bar{\theta} - \underline{\theta}\right) > \phi$ ensures that it is not optimal to always exclude one bidder, implying that it is optimal for one bidder to always enter, and for the other to enter with an ex-ante probability between zero and one. In this case, first-order conditions yield that the other potential bidder should enter when its valuation is at least $\underline{\theta} + \sqrt{\frac{2\phi(\bar{\theta}-\underline{\theta})}{2-\alpha}} < \bar{\theta}$.

Proposition 3 offers insights into the factors that affect the relative merits of symmetric and asymmetric mechanisms, as bidders earn strictly positive rents that depend on the royalty rate α and differ between these two types of mechanisms. Proposition 3 reveals that the benefit of the asymmetric mechanism rises with α.[11] Greater α make the condition

[11]The result that the value of an asymmetric mechanism rises with α extends to settings where $\underline{\theta}$ is low

$\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$ less stringent. While α does not affect the social welfare benefits of spreading, the proof reveals that spreading the entry thresholds increases total bidder payoffs at a seller's expense. Decreased tying (reduced α) raises bidder profits, magnifying this effect, especially when the entry threshold is low.[12] The conditions in the proposition ensure that the positive effect of spreading on social welfare outweighs the negative effect of increasing total bidder payoffs.

With entry costs, expected revenues in the optimal mechanism increase in α for two reasons. First, increasing the royalty rate α reduces the profit of the bidder who always enters. In turn, this reduces the value that the seller attaches to entry by the other bidder. Thus, second, the seller can further increase revenue by setting a higher entry threshold for the other bidder, i.e., increasing the degree of asymmetry in cutoffs.

Implementation costs for asymmetric mechanisms. Any optimal asymmetric mechanism that induces entry cutoffs $(\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt})$ with $\hat{\theta}_1^{opt} < \hat{\theta}_2^{opt}$ can be implemented by a second-price auction with reserve price $r_i = (1-\alpha)\hat{\theta}_i^{opt}$ for bidder $i \in \{1, 2\}$ and entry subsidy ϕ to any entrant i whose bid exceeds r_i, where the auction winner i makes the additional royalty payment αZ_i with $\alpha \in [0, 1)$.[13] These reserve prices and subsidy reflect that after entering the auction, entry costs are sunk, it is a weakly dominant strategy for a bidder with valuation θ to bid $(1-\alpha)\theta$ in the second-price auction with royalty rate α, and each entrant is reimbursed with its entry cost to address individual rationality.

In practice, asymmetric mechanisms may be costly to implement. For example, there may be legal costs of discriminating between bidders in auctions that feature asymmetric rules or complexity costs associated with inducing an asymmetric equilibrium among the multiple-equilibria in auctions that use symmetric rules.[14] To model this we add a quadratic

enough that $2\underline{\theta} - \bar{\theta} < \phi$ (so that **A1** is violated). However, when we fix α and decrease $\underline{\theta}$, the benefit of the asymmetric mechanism decreases: if $\underline{\theta}$ is reduced below zero, the asymmetric mechanism ceases to be optimal for all α (see equation (27) in the proof).

[12] By (8) in Lemma 1 and $\pi_j(\hat{\theta}_j) = 0$ in Lemma 2, bidder i's expected profit given signal θ_i is $(1-\alpha)\int_{\hat{\theta}_i}^{\theta_i} G_i(\theta)d\theta$, where $G_i(\theta)$ is the probability that i wins given signal θ. This profit decreases in α and $\hat{\theta}_i$.

[13] With ex-ante symmetric bidders, this mechanism corresponds to that used in the proof of Lemma 2, and as the proof of the lemma shows, it implements the optimal mechanism.

[14] Campbell (1998) provides a sufficient condition for the existence of asymmetric equilibria in second-price auctions with two bidders.

cost $\gamma(\hat{\theta}_2 - \hat{\theta}_1)^2$ for asymmetric mechanisms with $\hat{\theta}_1 \neq \hat{\theta}_2$.

Proposition 4 *Suppose **A1** holds, and there is a quadratic cost $\gamma(\hat{\theta}_2 - \hat{\theta}_1)^2$ for adopting mechanisms with $\hat{\theta}_2 \neq \hat{\theta}_1$. Then an asymmetric mechanism with $\hat{\theta}_1 \neq \hat{\theta}_2$ is optimal if and only if $\gamma < \gamma^*(\alpha)$, where*

$$\gamma^*(\alpha) \equiv \frac{2\underline{\theta} - \bar{\theta} + (\bar{\theta} - \underline{\theta})\alpha}{4(\bar{\theta} - \underline{\theta})^2}$$

increases in α.

One might posit that because the costs of slight separation are only of second order, some separation is always optimal under **A1**. However, this logic is flawed: the benefits of slight separation are also of second order. In fact, asymmetric mechanisms are optimal if and only if the asymmetry cost parameter γ is sufficiently small, i.e., $\gamma < \gamma^*(\alpha)$. $\gamma^*(\alpha)$ increases in α—with higher royalty rates, asymmetric mechanism can remain optimal even when the costs of implementing asymmetric mechanisms are higher. This result reinforces our earlier findings that the relative benefit of asymmetric mechanisms over symmetric ones increases in α.

4 Heterogeneous Bidders

The preceding analysis establishes how strategic considerations can lead a seller to treat bidders heterogeneously even when bidders are ex-ante identical so that the choice of which bidder to favor is arbitrary. In practice, ex ante, a seller is often aware of dimensions along which potential bidders differ. For example, a target firm may believe that one potential acquiring firm is more likely to have higher synergies than another. The target could reinforce the initial asymmetry by favoring the bidder with the higher expected synergies. Alternatively, it could encourage participation by the bidder with lower expected synergies by favoring it.

We now address how asymmetries in primitives affect a seller's choice of *which* bidder to favor, and how extensive such favoritism should be. That is, we derive how different forms of ex-ante bidder heterogeneity affect the optimal auction design. We first consider a setting where bidders have different entry costs ϕ_i, but are otherwise identical. We then consider bidders who have different supports for their valuation distributions F_i.[15]

[15] In our fixed royalty rate setting, project investment costs (as in DKS) do not affect outcomes, even if

4.1 Heterogeneity in entry costs

We first show that when bidders differ in ϕ_i, there is a natural bidder to favor—the low entry cost bidder—making the optimal way to select the favored bidder unique. Thus, in contrast to the costless entry setting of Myerson (1981), there is a *discontinuity* in the optimal degree of asymmetry with respect to the underlying bidder heterogeneity.

Assumption A1′: There are two potential bidders whose valuations are independently and identically uniformly distributed on $[\underline{\theta}, \bar{\theta}]$, but who have different entry costs $\phi_2 > \phi_1 > 0$, where $2\underline{\theta} - \bar{\theta} > \phi_1$ and $\frac{\bar{\theta}-\underline{\theta}}{2} > \phi_2$.

Proposition 5 *Under **A1′**, the optimal mechanism favors the low cost entrant:* $\hat{\theta}_1 < \hat{\theta}_2 < \bar{\theta}$.

The intuition is straightforward. Because bidder valuations are symmetrically distributed, when it is optimal to favor one bidder in the auction design, efficiency considerations mandate that the seller should favor the bidder with lower entry costs.

4.2 Heterogeneity in the distribution of bidder valuations

We now consider bidders who have the same lower support $\underline{\theta}$ for their valuations, but different upper supports. Without loss of generality, we suppose that $\bar{\theta}_1 > \bar{\theta}_2$. We again assume $2\underline{\theta} - \bar{\theta}_1 > 0$. We will show that with heterogeneous supports, the size of ϕ is critical for determining which bidder to favor and the extent. Accordingly, we place minimal restrictions on ϕ, requiring only that $\phi < \bar{\theta}_1$ ($\phi \geq \bar{\theta}_1$ is uninteresting because zero entry becomes optimal). With uniformly-distributed valuations, the virtual valuation (10) of bidder $i \in \{1, 2\}$ is

$$\begin{aligned} \xi_i(\theta_i) &= \theta_i - (1-\alpha)\left(\bar{\theta}_i - \theta_i\right) \\ &= (2-\alpha)\,\theta_i - (1-\alpha)\,\bar{\theta}_i. \end{aligned} \tag{17}$$

Bidder 1 corresponds to the “strong bidder” in Myerson (1981). Myerson shows that without entry costs, optimal selling mechanisms handicap this strong bidder. The logic is that if both bidders have the same valuation θ, then the strong bidder has a lower virtual valuation, so the

they differ between bidders. Such costs do matter in pure equity auctions, where investment costs affect a bidder’s information advantage and favoring bidders with smaller investment costs is optimal (Liu, 2016).

seller benefits from setting a higher reserve price for this bidder. Extending this to our costly entry setting, one might conjecture that it should be optimal to "favor" weak bidder 2 by setting a lower threshold for bidder 2. Counterintuitively, we establish that when entry is costly, the optimal bidder to favor varies, depending on both the entry costs and the royalty rate.

The logic reflects the insight highlighted earlier underlying how asymmetric thresholds affect the probability of no sale. We explain in two steps. As a starting point, consider a small spread away from an initial position with identical thresholds $\theta_1^* = \theta_2^* \in \left(\underline{\theta}, \min\left(\bar{\theta}_1, \bar{\theta}_2\right)\right)$, so that bidder i enters with probability $p_i = (\bar{\theta}_i - \theta_i^*)/(\bar{\theta}_i - \underline{\theta})$, to asymmetric entry thresholds $\hat{\theta}_1 = \theta_1^* - \epsilon^*$ and $\hat{\theta}_2 = \theta_2^* + \epsilon$, where ϵ and ϵ^* are chosen so that bidder 1 enters with probability $p_1 + \Delta p$ and bidder 2 with $p_2 - \Delta p$. Importantly, unlike the symmetric case, now $p_1 > p_2$ even though initial entry thresholds are identical, $\theta_1^* = \theta_2^*$. The probability of no sale becomes

$$[1 - (p_1 + \Delta p)][1 - (p_2 - \Delta p)] = (1 - p_1)(1 - p_2) - (\Delta p)^2 - (p_1 - p_2)\Delta p.$$

This equation conveys the key change introduced by heterogeneity in bidders' valuation distributions. The second term $-(\Delta p)^2$ is negative regardless of the sign of Δp; this reflects the advantage of introducing asymmetric cutoffs as in a symmetric setting. However, now, there is a third term, $-(p_1 - p_2)\Delta p$, which is negative only if $\Delta p > 0$. This means that when $p_1 > p_2$, increasing p_1 and decreasing p_2 is more effective at reducing the probability of no sale than increasing p_2 and decreasing p_1. Thus, favoring bidder 1 increases efficiency.

Next, recognize that the threshold choices θ_1^* and θ_2^* should reflect virtual valuations, not actual valuations. Heuristically, one can think of θ_1^* and θ_2^* as solving a constrained optimization problem: maximize a seller's expected revenue Π_s subject to an equal virtual valuation constraint, $\xi_1(\theta_1^*) = \xi_2(\theta_2^*) \equiv \xi^*$. With heterogeneity in the distribution of valuations, the solution typically features $\theta_1^* \neq \theta_2^*$, and, as we detail, p_1 may be either larger or smaller than p_2.

Thus, collectively, there are two consequences to favoring a "strong" bidder. First, Myerson's logic implies that favoring the stronger bidder reduces a seller's ability to extract informational rents. Because greater equity shares α reduce bidder information rents, this force declines when α increases. Second, there is an efficiency effect, the direction of which depends on whether $p_1 > p_2$ or $p_1 < p_2$. Recall that θ_1^* and θ_2^* solve the constrained optimization problem. Therefore, ξ^* increases in ϕ. In a benchmark setting where ϕ approaches 0, entry becomes almost costless, and Myerson's logic implies that the optimal mechanism features $\theta_1^* > \theta_2^* = \underline{\theta}$: because $\xi_2(\underline{\theta}) > 0$ by $2\underline{\theta} > \bar{\theta}_i$, the optimal design features $\theta_2^* = \underline{\theta}$ and $\xi_1(\theta_1^*) = \xi_2(\underline{\theta})$.[16] As a result, $p_1 < p_2 = 1$. When ϕ is positive but small, the efficiency considerations embodied in the third term "$-(p_1 - p_2)\,\Delta p$" reinforce the Myerson logic, i.e., it becomes optimal to increase $\hat{\theta}_1$ even further above $\hat{\theta}_2$, beyond the degree when entry is costless. However, because ξ^* increases in ϕ and $\xi_1(\bar{\theta}_1) > \xi_2(\bar{\theta}_2)$, as ϕ is increased, p_1 eventually reaches p_2. Beyond that point, further increases in ϕ raise p_1 past p_2, making it more effective to reduce the probability of no sale by closing the gap between $\hat{\theta}_1$ and $\hat{\theta}_2$. When this efficiency effect is strong enough to overcome the Myerson logic (which falls with α), the difference $\hat{\theta}_1 - \hat{\theta}_2$ not only shrinks, but it reverses sign: the optimal design can feature $\hat{\theta}_1 < \hat{\theta}_2$.

For simplicity, we first consider the limiting case where $\bar{\theta}_2$ approaches $\underline{\theta}$, i.e., there is little uncertainty about the weak bidder's valuation. This yields a closed-form solution that demonstrates how the two forces play out. We then illustrate numerically how the intuition extends when $\bar{\theta}_2 - \underline{\theta}$ is non-negligible. To ease presentation, we work with the probability of entry rather than the threshold of entry for the weak bidder 2, defining $p_2 \equiv \frac{\bar{\theta}_2 - \hat{\theta}_2}{\bar{\theta}_2 - \underline{\theta}}$ to be the probability that the bidder enters, i.e., $\hat{\theta}_2 = \underline{\theta} + (1 - p_2)\,(\bar{\theta}_2 - \underline{\theta})$. We have:

Proposition 6 *Assume uniform distribution, $2\underline{\theta} - \bar{\theta}_1 > 0$ and $\phi < \bar{\theta}_1$. Consider the case where $\bar{\theta}_2$ approaches $\underline{\theta}$. Then there exists a cutoff value $\phi^*(\alpha) \in \left(0, \bar{\theta}_1 - \underline{\theta}\right)$ (defined in the proof), with $\frac{d\phi^*(\alpha)}{d\alpha} < 0$ such that*

(i) If $\phi \in (0, \phi^*(\alpha))$, then the weak bidder always enters, $p_2 = 1$, and

$$\hat{\theta}_1 = \frac{(1-\alpha)\,\bar{\theta}_1 + \underline{\theta} + \phi}{2-\alpha} > \underline{\theta}. \tag{18}$$

$\hat{\theta}_1$ strictly decreases in α, and strictly increases in ϕ ($\frac{\partial \hat{\theta}_1}{\partial \phi} > 0$), where the rate of increase grows with the royalty rate α ($\frac{\partial^2 \hat{\theta}_1}{\partial \alpha \partial \phi} > 0$).

[16] In the special case of costless entry, $\phi = 0$, the value of θ_1^* is not unique: any value below the solution of $\xi_1(\theta_1^*) = \xi_2(\underline{\theta})$, e.g., $\theta_1^* = \underline{\theta}$, can also be optimal, even though bidder types with values below $\xi_1(\theta_1^*) = \xi_2(\underline{\theta})$ have no chance of winning. This non-uniqueness reflects that when $\phi = 0$, social welfare is not wasted by entry of bidder types with no chance of winning. Once ϕ is positive, it is not optimal to have such bidder types enter.

(ii) If $\phi \in \left(\phi^*(\alpha), \bar{\theta}_1\right)$ *and* $\alpha > 0$,[17] *then the weak bidder never enters,* $p_2 = 0$*, and*

$$\hat{\theta}_1 = \begin{cases} \underline{\theta} & \text{if} \quad \phi \leq (2-\alpha)\underline{\theta} - (1-\alpha)\bar{\theta}_1 \\ \frac{(1-\alpha)\bar{\theta}_1 + \phi}{(2-\alpha)} & \text{if} \quad \phi > (2-\alpha)\underline{\theta} - (1-\alpha)\bar{\theta}_1, \end{cases} \tag{19}$$

which weakly decreases in α*.*

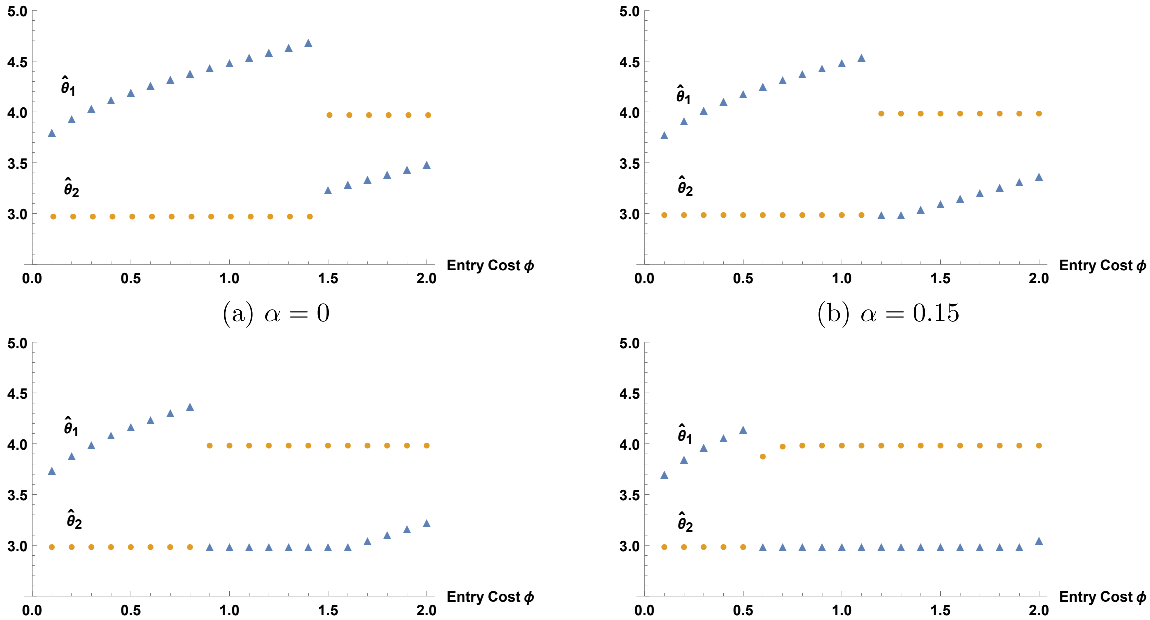
(iii) $\hat{\theta}_1$ *is nonmonotone in* ϕ*: it decreases as* ϕ *crosses* $\phi^*(\alpha)$*,* $\lim_{\phi \to \phi^*(\alpha)^-} \hat{\theta}_1 > \lim_{\phi \to \phi^*(\alpha)^+} \hat{\theta}_1$*, but it weakly increases in* ϕ*,* $\frac{\partial \hat{\theta}_1}{\partial \phi} \geq 0$ *for* $\phi > \phi^*(\alpha)$*.*

These results show that in the limit as ϕ approaches 0, the logic of the costless Myerson benchmark obtains, so that $\hat{\theta}_2 = \underline{\theta}$ and $\hat{\theta}_1 > \underline{\theta}$. Then, as the entry cost ϕ rises, $\hat{\theta}_2$ remains at $\underline{\theta}$, but $\hat{\theta}_1$ continuously increases, where the rate at which $\hat{\theta}_1$ increases in ϕ rises when the royalty rate α is higher. Eventually, ϕ reaches $\phi^*(\alpha)$, at which point $\hat{\theta}_1$ *falls* to $\underline{\theta}$ and the probability that bidder 2 enters falls to 0, i.e., $\hat{\theta}_2 = \bar{\theta}_2$. Once ϕ reaches a second-higher threshold, $\max\left\{\phi^*(\alpha), (2-\alpha)\underline{\theta} - (1-\alpha)\bar{\theta}_1\right\}$, $\hat{\theta}_1$ increases again while the probability that bidder 2 enters remains at 0. Finally, increases in α *reduce* the optimal magnitude of asymmetry when ϕ is small, but *increase* it when ϕ is large: $\hat{\theta}_1$ decreases in α regardless of whether $\phi < \phi^*(\alpha)$ or $\phi > \phi^*(\alpha)$; but $p_2 = 1$ when the entry cost is small, and $p_2 = 0$ when the entry cost is large.

These findings reflect the intuition highlighted earlier. First, when entry costs are small, it is optimal to magnify the threshold difference in the direction of the Myerson logic, further favoring the weak bidder—beyond the degree when entry is costless. However, once entry costs are sufficiently large, the reverse—favoring a "strong" bidder—becomes optimal. Further, the relative importance of efficiency considerations—as embodied in the term "$-(p_1 - p_2)\Delta p$"—increases with ϕ (i.e., $p_1 - p_2$ becomes large when the entry cost is high), and as entry cost rises, there is a "regime change" when the efficiency consideration becomes more important than the rent-extraction (Myerson) consideration. Moreover, for any given entry cost, the relative importance of rent-extraction considerations decreases in α because higher α reduces bidder rents. It follows that when entry costs are low, greater royalty rates make it optimal to favor the weaker bidder by less, reducing the optimal degree of asymmetry; but when entry costs are high, greater royalty rates make it optimal to favor the

[17] Even when $\alpha = 0$ and $\phi > \phi^*(\alpha)$ or α is arbitrary and $\phi = \phi^*(\alpha)$, $p_2 = 0$ and (19) are still a solution for the optimization. However, other solutions may also exist.

stronger bidder by more, raising the optimal degree of asymmetry.



(a) $\alpha = 0$

(b) $\alpha = 0.15$

(c) $\alpha = 0.30$

(d) $\alpha = 0.45$

Figure 2: Optimal entry cutoffs $\hat{\theta}_1$ and $\hat{\theta}_2$ as a function of entry cost ϕ for different royalty rates α with independently uniformly distributed valuations, over $[3, 5]$ for bidder 1 and $[3, 4]$ for bidder 2. That is, bidder 1 is stronger than bidder 2.

Figure 2 illustrates how the key ideas in Proposition 6 extend to settings in which $\bar{\theta}_2 - \underline{\theta}$ is non-negligible. It plots optimal cutoffs for a strong bidder 1 whose valuation θ_1 is drawn from a uniform distribution on $[3, 5]$ and a weak bidder 2 whose valuation θ_2 is drawn from a uniform distribution on $[3, 4]$, for four fixed royalty rates α. The figure indicates that it is optimal to favor the weak bidder when the entry cost ϕ is small: $\hat{\theta}_1 > \hat{\theta}_2 = \underline{\theta}$. As ϕ rises, $\hat{\theta}_2$ remains at $\underline{\theta}$, but $\hat{\theta}_1$ continuously increases. However, eventually ϕ hits a critical value at which the optimal entry cutoffs discontinuously reverse so that $\hat{\theta}_1 < \hat{\theta}_2$. That is, once ϕ exceeds a critical value, it becomes optimal to favor the strong bidder. This critical value decreases in α reflecting that the relative importance of efficiency concerns increases with α.[18] Further, increases in α make it optimal to favor a weak bidder by less, i.e., $\hat{\theta}_1 - \hat{\theta}_2$ shrinks when entry costs are low; but to favor a strong bidder by more, i.e., $\hat{\theta}_2 - \hat{\theta}_1$ expands when entry cost are high.

[18]These patterns of reversal and nonmonotonicity with respect to entry costs are unique features of settings with heterogeneous bidders, where, as Proposition 6 and the example show, at each ϕ there is typically a unique optimum. With ex-ante identical bidders, at each ϕ there are typically two equilibria (one with $\hat{\theta}_1 < \hat{\theta}_2$ and the other with $\hat{\theta}_1 > \hat{\theta}_2$) and no transition can be discerned.

Collectively, these findings establish the robustness of our results beyond small deviations from the baseline model, and show that the consequences are non-trivial for large deviations. The optimal bidder to favor and the extent of such favoritism depends on the royalty rate and the size of entry costs in non-monotonic and discontinuous ways.

5 Conclusion

With entry costs, optimal selling mechanisms trade off between the increased rents that more entrants can bring and the higher total entry costs incurred by more bidders that a seller indirectly bears via the endogenous entry choices. We establish that as long as there is enough uncertainty over bidder valuations relative to entry costs, it is optimal to handicap some bidders in order to encourage other bidders to enter. When bidders are ex-ante identical, greater royalty rates differentially reduce the profit of a bidder who enters more frequently, making it optimal to raise the reserve for another bidder.

In practice, bidders typically differ ex ante in their attributes. We derive how the nature of ex-ante bidder heterogeneity affects a seller's choice of which bidder, if any, to favor, and how extensively to favor that bidder. When bidders differ in their participation costs, efficiency considerations lead a seller to favor the low entry cost bidder. Optimal designs are more complex when a seller knows that one bidder is likely to have a higher valuation than another. The rent extraction considerations present in Myerson's (1981) costless entry setting provide incentives to favor the weaker bidder in order to extract more informational rents from the stronger bidder when its valuation is high. However, a seller also wants to minimize the probability of no sale. If entry costs are low, the weaker bidder is more likely to enter so the seller favors the weaker bidder beyond the degree prescribed by Myerson, thereby reducing the probability of no sale. However, as entry costs rise, so do optimal entry cutoffs, and eventually the stronger bidder becomes more likely to enter. Once entry costs rise by enough, it becomes optimal to favor the stronger bidder. Greater royalty rates reduce the optimal degree of asymmetry if entry costs are low, but increase the asymmetry if entry costs are high.

Our analysis provides theoretical foundations for the asymmetric auction designs of procurement auctions that favor certain designated suppliers, and the designs of wireless spectrum auctions that favor particular types of bidders. So, too, target firms in takeover auctions frequently design bidding rules that favor one bidder over another (see, e.g., Povel and Singh, 2006). For example, a target often elicits an initial bid from one bidder by promising to reimburse it with a break-up fee that compensates the bidder for its efforts. This commits the target firm to excluding other bidders unless their valuations are sufficiently high.

In the appendix, we contrast our setting with one in which potential bidders only learn valuations *after* entering. We establish that when potential bidders do not have an information advantage over the seller when making entry decisions, the seller can extract all bidder rents by using lump-sum transfers: tying payments to valuations is unnecessary. Moreover, the seller need not handicap particular bidders when bidders are ex-ante identical—the optimal amount of entry arises endogenously in equilibrium.

Appendix

Proof of Lemma 1: By (1) and (4), $\pi_i(\cdot)$ is the maximum of a family of affine functions, so it is convex. Because $\pi_i(\cdot)$ is also bounded and hence absolutely continuous, it is differentiable almost everywhere in the interior of its domain. By (7), at all $\theta_i' > \hat{\theta}_i$, a line at θ_i' with a slope of $(1-\alpha)G_i(\theta_i')$ supports the function $\pi_i(\cdot)$. Thus, at each point that $\pi_i(\cdot)$ is differentiable, $\frac{d\pi_i(\theta_i)}{d\theta_i} = (1-\alpha)G_i(\theta_i)$. The first part of the lemma follows because an absolutely continuous function is the definite integral of its derivative.

The seller's expected revenues (6) can be rewritten as

$$
\begin{aligned}
&\sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \int_{\theta_{-i}} [T_i(\theta_i, m_{-i}(\theta_{-i})) + W_i(\theta_i, m_{-i}(\theta_{-i})) \cdot \alpha\theta_i] f_{-i}(\theta_{-i})\, d\theta_{-i} dF_i(\theta_i) \\
&= \sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \left[\int_{\theta_{-i}} T_i(\theta_i, m_{-i}(\theta_{-i})) f_{-i}(\theta_{-i})\, d\theta_{-i} + G_i(\theta_i) \cdot \alpha\theta_i \right] dF_i(\theta_i) \\
&= \sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \left[G_i(\theta_i)\theta_i - \int_{\hat{\theta}_i}^{\theta_i} (1-\alpha)G_i(t)dt - \pi_i(\hat{\theta}_i) - \phi_i \right] dF_i(\theta_i) \\
&= \sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \left[G_i(\theta_i)\theta_i - \int_{\hat{\theta}_i}^{\theta_i} (1-\alpha)G_i(t)dt \right] dF_i(\theta_i) - \sum_i (1 - F_i(\hat{\theta}_i)) \left(\pi_i(\hat{\theta}_i) + \phi_i \right),
\end{aligned}
$$

where the second equality follows from substituting for $\int_{\theta_{-i}} T_i(\theta_i, m_{-i}(\theta_{-i})) f_{-i}(\theta_{-i})\, d\theta_{-i}$ using (1) and then substituting for $\pi_i(\theta_i)$ using (8). Further,

$$
\int_{\hat{\theta}_i}^{\overline{\theta}_i} \int_{\hat{\theta}_i}^{\theta_i} (1-\alpha) G_i(t) dt dF_i(\theta_i) = \int_{\hat{\theta}_i}^{\overline{\theta}_i} (1-\alpha)(1 - F_i(\theta_i)) G_i(\theta_i)\, d\theta_i
$$

follows from integrating by parts, so the seller's expected revenue can be rewritten as

$$
\begin{aligned}
&\sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \left[\theta_i - (1-\alpha) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] G_i(\theta_i) dF_i(\theta_i) - \sum_i (1 - F_i(\hat{\theta}_i)) \left(\pi_i(\hat{\theta}_i) + \phi_i \right) \\
&= \sum_i \int_{\hat{\theta}_i}^{\overline{\theta}_i} \int_{\theta_{-i}} \xi_i(\theta_i; \alpha) W_i(\theta_i, m_{-i}(\theta_{-i})) f(\theta)\, d\theta - \sum_i (1 - F_i(\hat{\theta}_i)) \left(\pi_i(\hat{\theta}_i) + \phi_i \right) \\
&= \sum_i \int_{\theta} \xi_i(\theta_i; \alpha) W_i(m(\theta)) f(\theta)\, d\theta - \sum_i (1 - F_i(\hat{\theta}_i)) \left(\pi_i(\hat{\theta}_i) + \phi_i \right).
\end{aligned}
$$

The first equality follows from substituting (2) for $G_i(\theta_i)$ and the definition of $\xi_i(\theta_i; \alpha)$ in (10). The last equality holds because $W_i(m(\theta)) = 0$ if $m_i(\theta_i) = \emptyset$ and $m_i(\theta_i) = \emptyset$ for $\theta_i < \hat{\theta}_i$. $\square$

Proof of Lemma 2: We first show that the expected revenue in (11) is attainable. Consider the following mechanism: Given reports $m \in M$, let W be such that the seller gives the asset to the bidder who reports $m_i \neq \emptyset$ and has the highest $\xi_i(\theta_i; \alpha)$ such that $m_i = \theta_i \geq \hat{\theta}_i$. For any given θ_{-i}, denote the infimum of all bids such that bidder i can win against θ_{-i} by

$$z_i(\theta_{-i}) \equiv \inf\{\theta_i \geq \hat{\theta}_i : \xi_i(\theta_i; \alpha) \geq \xi_j(\theta_j; \alpha), \ \forall j \in \{k : \theta_k \geq \hat{\theta}_k\}\}$$

and, for any given reports $m \in M$, let the payment rule be

$$T_i(m) = \begin{cases} (1-\alpha)\, z_i(\theta_{-i}) - \phi_i & \text{if bidder } i \text{ wins} \\ -\phi_i & \text{if } m_i = \theta_i \geq \hat{\theta}_i \text{ and bidder } i \text{ loses} \\ 0 & \text{if } m_i = \emptyset \text{ or } m_i < \hat{\theta}_i. \end{cases}$$

We show that truth-telling is an equilibrium. For type $\theta_i < \hat{\theta}_i$, it is optimal to report $\emptyset$ and not enter because, if he instead entered and bids above $\hat{\theta}_i$ (so that his entry cost could be reimbursed), his winning profit would be negative. Next, suppose that type $\theta_i \geq \hat{\theta}_i$ reports $\theta_i' > \theta_i$ instead of θ_i. If $z_i(\theta_{-i}) > \theta_i' > \theta_i$, then bidder i still loses; if $\theta_i' \geq z_i(\theta_{-i}) > \theta_i$, then bidder i wins and makes a negative profit of $(1-\alpha)\theta_i - \phi_i - ((1-\alpha)z_i(\theta_{-i}) - \phi_i) = (1-\alpha)(\theta_i - z_i(\theta_{-i}))$ rather than the zero profit it would have received from bidding θ_i; and if $\theta_i' > \theta_i \geq z_i(\theta_{-i})$, then bidder i still wins and makes the same profit. Suppose that type $\theta_i \geq \hat{\theta}_i$ reports $\theta_i' < \theta_i$ instead of θ_i. If $z_i(\theta_{-i}) \geq \theta_i > \theta_i'$, then bidder i still loses; if $\theta_i > z_i(\theta_{-i}) > \theta_i'$, then bidder i loses and makes a zero profit instead of making the positive profit of $(1-\alpha)(\theta_i - z_i(\theta_{-i}))$ that it would have received from bidding θ_i; and if $\theta_i > \theta_i' \geq z_i(\theta_{-i})$, then bidder i still wins and makes the same profit. Thus, truth-telling is an equilibrium. Further, because $\pi_i(\hat{\theta}_i) = 0$ for all i in this mechanism, (9) reduces to (11), proving that (11) is attainable.

Observe that the first term on the right-hand side of (9) cannot exceed that of (11) (the equality holds if and only if the first part of the lemma is satisfied), and neither does the second term by $\pi_i(\hat{\theta}_i) \geq 0, \forall i$. Thus, (11) is the expected revenue in the optimal mechanism. □

Proof of Proposition 1: With a single potential bidder and any α, a seller can always extract a surplus of at least $\bar{\theta} - \epsilon - \phi$ by making a take-it-or-leave-it cash demand of $(1-\alpha)(\bar{\theta} - \epsilon) - \phi$, together with the royalty payment associated with the fixed rate α. This leaves the lowest valuation type $\bar{\theta} - \epsilon$ indifferent to entry. Hence, $\Pi(p, n = 1) \geq \bar{\theta} - \epsilon - \phi$.

Now consider $n > 1$ potential bidders. If the probability that each bidder enters is p, the seller's payoff is bounded as follows:

$$\Pi\left(p, n > 1\right) < \left(1 - \left(1 - p\right)^n\right) \bar{\theta} - np\phi, \tag{20}$$

where $1 - (1 - p)^n$ is the probability that at least one potential bidder enters. Thus, $\left(1 - \left(1 - p\right)^n\right) \bar{\theta}$ is an upper bound on welfare gains and $np\phi$ is the expected entry cost (the inequality is slack because the winner's type is typically below $\bar{\theta}$ and bidders may earn positive rents). Maximizing the right-hand side of (20) with respect to p yields $p^* = 1 - (\frac{\phi}{\bar{\theta}})^{\frac{1}{n-1}}$. Substituting this into the right-hand side of (20) yields

$$\Pi\left(p, n > 1\right) < \bar{\theta} - n\phi + (n-1)\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}} \phi.$$

One can show that the right-hand side decreases in n for $n \geq 2$. Thus,

$$\Pi\left(p, n > 1\right) < \bar{\theta} - 2\phi + \frac{\phi^2}{\bar{\theta}} < \bar{\theta} - \varepsilon - \phi \leq \Pi\left(p, n = 1\right),$$

where the second inequality follows from $\epsilon < \phi\left(1 - \frac{\phi}{\bar{\theta}}\right)$. $\square$

Proof of Proposition 2: Explicitly writing out the max term in (16) yields that, for $\hat{\theta}_1 \leq \hat{\theta}_2$,

$$\begin{aligned}\Pi^*(\hat{\theta}_1, \hat{\theta}_2) = &\int_{\hat{\theta}_2}^{\bar{\theta}} \left(\int_{\hat{\theta}_1}^{\theta_2} \theta_2 dF\left(\theta_1\right) + \int_{\theta_2}^{\bar{\theta}} \theta_1 dF\left(\theta_1\right) \right) dF\left(\theta_2\right) \\ &+ F(\hat{\theta}_2) \int_{\hat{\theta}_1}^{\bar{\theta}} \theta_1 dF\left(\theta_1\right) + F(\hat{\theta}_1) \int_{\hat{\theta}_2}^{\bar{\theta}} \theta_2 dF\left(\theta_2\right) - \phi(2 - F(\hat{\theta}_1) - F(\hat{\theta}_2)).\end{aligned} \tag{21}$$

We first show that if $\phi < \bar{\theta} - E\left[\theta\right]$, then always excluding a bidder (without loss of generality bidder 2) is not optimal. If only bidder 1 enters then because $\phi < \underline{\theta}$, setting $\hat{\theta}_1 = \underline{\theta}$ is optimal. Differentiating (21) at $(\hat{\theta}_1 = \underline{\theta}, \hat{\theta}_2 = \bar{\theta})$ with respect to $\hat{\theta}_2$ yields

$$\left.\frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_2}\right|_{\hat{\theta}_1 = \underline{\theta},\, \hat{\theta}_2 = \bar{\theta}} = -f\left(\bar{\theta}\right)\left(\bar{\theta} - E[\theta] - \phi\right) < 0, \tag{22}$$

where the inequality holds by $\bar{\theta} - E\left[\theta\right] > \phi$. Thus, always excluding a bidder is not optimal.

We now show that symmetric mechanisms with $\hat{\theta}_1 = \hat{\theta}_2$ are never optimal. Because always excluding a bidder and always having two bidders (i.e., setting $\hat{\theta}_1 = \hat{\theta}_2 = \underline{\theta}$) are never optimal, without loss of generality suppose that $\hat{\theta}_1 \leq \hat{\theta}_2 \in \left(\underline{\theta}, \bar{\theta}\right)$. Consider the asymmetric mechanism: $\hat{\theta}_2 = \theta^* + \epsilon$ and $\hat{\theta}_1 = \theta^* - \epsilon^*$, where ϵ is small and ϵ^* solves $F\left(\theta^*\right) - F\left(\theta^* - \epsilon^*\right) = F\left(\theta^* + \epsilon\right) - F\left(\theta^*\right) \equiv \Delta p$. Using "$o$" to stand for "terms of order," we have

$$\Delta p = f\left(\theta^*\right)\epsilon + o\left(\epsilon^2\right) \quad \text{and} \quad \epsilon^* = \epsilon + o\left(\epsilon^2\right).$$

We show that $\Delta\Pi^* = \Pi^*\left(\theta^* - \epsilon^*, \theta^* + \epsilon\right) - \Pi^*\left(\theta^*, \theta^*\right) > 0$. We need only compare terms concerning social welfare because ϵ^* is set to equate the terms with ϕ in both mechanisms. We retain terms up to order ϵ^2. There exist contributions to $\Delta\Pi^*$ only in 3 cases:

Case 1: $\theta_2 \in \left(\theta^*, \theta^* + \epsilon\right)$ and $\theta_1 \in \left(\theta^* - \epsilon^*, \theta^*\right)$. The contribution to $\Delta\Pi^*$ is

$$-\left(\Delta p\right)^2\left(\epsilon + o\left(\epsilon^2\right)\right) = 0 + o\left(\epsilon^3\right).$$

Case 2: $\theta_2 \in \left(\theta^*, \theta^* + \epsilon\right)$ and $\theta_1 \notin \left(\theta^* - \epsilon^*, \theta^*\right)$. The contribution exists only when $\theta_1 \in \left(\underline{\theta}, \theta^* - \epsilon^*\right)$ and it is

$$\begin{aligned} -\Delta p\left(F\left(\theta^*\right) - \Delta p\right)\left(\theta^* + \frac{\epsilon}{2} + o\left(\epsilon^2\right)\right) &= -\Delta p\left(F\left(\theta^*\right) - \Delta p\right)\left(\theta^* + \frac{\epsilon}{2}\right) + o\left(\epsilon^3\right) \\ &= -\Delta p F\left(\theta^*\right)\left(\theta^* + \frac{\epsilon}{2}\right) + \left(\Delta p\right)^2\theta^* + o\left(\epsilon^3\right). \end{aligned}$$

Case 3: $\theta_2 \notin \left(\theta^*, \theta^* + \epsilon\right)$ and $\theta_1 \in \left(\theta^* - \epsilon^*, \theta^*\right)$. The contribution exists only when $\theta_2 \in \left(\underline{\theta}, \theta^*\right)$ and it is

$$\Delta p F\left(\theta^*\right)\left(\theta^* - \frac{\epsilon}{2} + o\left(\epsilon^2\right)\right) = \Delta p F\left(\theta^*\right)\left(\theta^* - \frac{\epsilon}{2}\right) + o\left(\epsilon^3\right).$$

Adding up all contributions from the 3 cases yields

$$\begin{aligned} \Delta\Pi^* =& \Delta p F\left(\theta^*\right)\left(\theta^* - \frac{\epsilon}{2}\right) - \Delta p F\left(\theta^*\right)\left(\theta^* + \frac{\epsilon}{2}\right) + \left(\Delta p\right)^2\theta^* + o\left(\epsilon^3\right) \\ =& \Delta p\left(\Delta p\theta^* - F\left(\theta^*\right)\epsilon\right) + o\left(\epsilon^3\right) \\ =& \Delta p\epsilon\left(f\left(\theta^*\right)\theta^* - F\left(\theta^*\right)\right) + o\left(\epsilon^3\right). \end{aligned}$$

Thus, symmetric mechanisms are never optimal if

$$f\left(\theta^*\right)\theta^* - F\left(\theta^*\right) > 0, \text{ for all } \theta^* \in \left(\underline{\theta}, \bar{\theta}\right), \tag{23}$$

or equivalently if $\frac{F(\theta^*)}{f(\theta^*)\theta^*} < 1$. We next show that (23) holds. Define $k \equiv \frac{1}{(\bar{\theta}-\underline{\theta})} \ln \frac{\bar{\theta}}{(\bar{\theta}-\underline{\theta})}$. The premise $\frac{df}{f} > -k$ implies that, for all $\theta \in [\underline{\theta}, \theta^*]$, $\frac{\ln f(\theta^*)-\ln f(\theta)}{\theta^*-\theta} > -k$, or $\ln \frac{f(\theta)}{f(\theta^*)} < k(\theta^* - \theta)$, which yields $f(\theta) < f(\theta^*) \exp(k(\theta^* - \theta))$. Thus,

$$\frac{F(\theta^*)}{f(\theta^*)\theta^*} < \frac{1}{\theta^*} \int_{\underline{\theta}}^{\theta^*} \exp(k(\theta^* - \theta))\, d\theta = \frac{\exp(k(\theta^* - \underline{\theta})) - 1}{k\theta^*}.$$

By the mean value theorem, there exists a $\theta^{**} \in [\underline{\theta}, \theta^*]$ such that $\exp(k(\theta^* - \underline{\theta})) - 1 = k \exp(k(\theta^{**} - \underline{\theta}))(\theta^* - \underline{\theta})$. Thus,

$$\begin{aligned} \frac{\exp(k(\theta^* - \underline{\theta})) - 1}{k\theta^*} &\le \frac{k \exp(k(\theta^* - \underline{\theta}))(\theta^* - \underline{\theta})}{k\theta^*} \\ &= \exp(k(\theta^* - \underline{\theta})) \frac{(\theta^* - \underline{\theta})}{\theta^*} \\ &\le \exp\left(k\left(\bar{\theta} - \underline{\theta}\right)\right) \frac{\left(\bar{\theta} - \underline{\theta}\right)}{\bar{\theta}} = 1. \end{aligned}$$

Thus, $\frac{F(\theta^*)}{f(\theta^*)\theta^*} < 1$, establishing (23). □

Proof of Proposition 3: We compare mechanism 1 in which $\underline{\theta} < \hat{\theta}_1 \le \hat{\theta}_2 < \bar{\theta}$, with mechanism 2 where $\hat{\theta}_1$ is replaced by $\hat{\theta}_1 - \epsilon$ and $\hat{\theta}_2$ replaced by $\hat{\theta}_2 + \epsilon$. Expected seller revenues equal expected social welfare minus expected bidder payoffs. Accordingly, we first calculate the difference in social welfare gains and then calculate the difference in expected bidder payoffs.

We first calculate the difference in social welfare gains $\Delta\Pi^* = \Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) - \Pi^*(\hat{\theta}_1, \hat{\theta}_2)$, where Π^* denotes social welfare defined in equation (21). Observe that there is a contribution to $\Delta\Pi^*$ only when both θ_1 and θ_2 lie in $[\underline{\theta}, \hat{\theta}_2 + \epsilon]$, which occurs with probability $\left(\frac{\hat{\theta}_2+\epsilon-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^2$. Consequently, with an abuse of notation, we compute contributions to $\Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon)$ and $\Pi^*(\hat{\theta}_1, \hat{\theta}_2)$ only when θ_1 and θ_2 are in $[\underline{\theta}, \hat{\theta}_2 + \epsilon]$. Replacing $\bar{\theta}$ with $\hat{\theta}_2 + \epsilon$ in equation (21), we have for mechanism 1

$$\begin{aligned} \Pi^*\left(\hat{\theta}_1, \hat{\theta}_2\right) = \left(\frac{\hat{\theta}_2 + \epsilon - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right)^2 \Bigg\{ \left(\frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}}\right)^2 \left(\hat{\theta}_2 + \frac{2}{3}\epsilon\right) + \frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 - \underline{\theta}}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \left(\hat{\theta}_2 + \frac{1}{2}\epsilon\right) \\ + \left(1 - \frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}}\right) \frac{\hat{\theta}_2 - \hat{\theta}_1 + \epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 + \hat{\theta}_1 + \epsilon}{2} \Bigg\} - \phi\left(\frac{2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1}{\bar{\theta} - \underline{\theta}}\right). \end{aligned}$$

The first term inside the braces corresponds to $\theta_2, \theta_1 \in [\hat{\theta}_2, \hat{\theta}_2 + \epsilon]$; the second term corresponds to $\theta_2 \in [\hat{\theta}_2, \hat{\theta}_2 + \epsilon]$ and $\theta_1 \in [\underline{\theta}, \hat{\theta}_2]$, and the third term corresponds to $\theta_2 \in [\underline{\theta}, \hat{\theta}_2]$ and

$\theta_1 \in [\hat{\theta}_1, \hat{\theta}_2 + \epsilon]$. For mechanism 2,

$$\Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) = \left(\frac{\hat{\theta}_2 + \epsilon - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{\frac{\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 + \hat{\theta}_1}{2}\right\} - \phi\left(\frac{2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1}{\bar{\theta} - \underline{\theta}}\right).$$

Thus,

$$\begin{aligned}
\Delta\Pi^* = & \left(\frac{1}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{\frac{1}{2}\left(\hat{\theta}_2 + \epsilon - \underline{\theta}\right)\left(\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon\right)\left(\hat{\theta}_2 + \hat{\theta}_1\right) - \epsilon^2\left(\hat{\theta}_2 + \frac{2}{3}\epsilon\right)\right. \\
& \left. - \epsilon\left(\hat{\theta}_2 - \underline{\theta}\right)\left(\hat{\theta}_2 + \frac{1}{2}\epsilon\right) - \frac{1}{2}\left(\hat{\theta}_2 - \underline{\theta}\right)\left(\hat{\theta}_2 - \hat{\theta}_1 + \epsilon\right)\left(\hat{\theta}_2 + \hat{\theta}_1 + \epsilon\right)\right\} \\
= & \left(\frac{1}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{-\frac{2}{3}\epsilon^3 + \left(\hat{\theta}_1 - \hat{\theta}_2 + \underline{\theta}\right)\epsilon^2 + \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left(\underline{\theta} - \frac{1}{2}\left(\hat{\theta}_2 - \hat{\theta}_1\right)\right)\epsilon\right\}. \qquad (24)
\end{aligned}$$

The leading term in (24) is ϵ whose coefficient is $(\hat{\theta}_2 - \hat{\theta}_1)(\underline{\theta} - \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1))$. As long as $\underline{\theta} > \frac{1}{3}\bar{\theta}$, which is implied by $2\underline{\theta} > \bar{\theta} + \phi$, this coefficient is positive, meaning that social welfare Π^* increases as the gap between the two cutoffs (slightly) widens.

We next calculate bidders' payoffs in the optimal mechanism with entry cutoffs $(\hat{\theta}_1, \hat{\theta}_2)$. By Lemmas 1 and 2, bidder i with $\hat{\theta}_i$ earns zero, the higher valuation bidder wins if both bidders enter, and a solo entrant wins. Thus, by (8), bidder i's unconditional expected profit is

$$\begin{aligned}
\pi_i &= (1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}}\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)\, d\theta dF(\theta_i) \\
&= -(1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}}\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)\, d\theta d(1 - F(\theta_i)) \\
&= (1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}} (1 - F(\theta_i)) W_i(\theta_i)\, d\theta_i.
\end{aligned}$$

Because $\hat{\theta}_1 \leq \hat{\theta}_2$, we have $W_2(\theta_2) = F(\theta_2)$ and

$$W_1(\theta_1) = \begin{cases} F(\theta_1) & \text{if} \quad \theta_1 > \hat{\theta}_2 \\ F(\hat{\theta}_2) & \text{if} \quad \theta_1 \in [\hat{\theta}_1, \hat{\theta}_2]. \end{cases}$$

Thus,

$$\begin{aligned}
\pi_1 &= (1-\alpha) F(\hat{\theta}_2)\int_{\hat{\theta}_1}^{\hat{\theta}_2} (1 - F(\theta_1))\, d\theta_1 + (1-\alpha)\int_{\hat{\theta}_2}^{\bar{\theta}} (1 - F(\theta_1)) F(\theta_1)\, d\theta_1 \\
\pi_2 &= (1-\alpha)\int_{\hat{\theta}_2}^{\bar{\theta}} (1 - F(\theta_2)) F(\theta_2)\, d\theta_2.
\end{aligned}$$

Letting $\pi^*(\hat{\theta}_1, \hat{\theta}_2)$ be the sum of both bidders' equilibrium payoffs,

$$\begin{aligned}\pi^*(\hat{\theta}_1, \hat{\theta}_2) &\equiv \pi_1 + \pi_2 \\ &= (1-\alpha) F\left(\hat{\theta}_2\right) \int_{\hat{\theta}_1}^{\hat{\theta}_2} (1 - F(\theta_1))\, d\theta_1 + 2(1-\alpha) \int_{\hat{\theta}_2}^{\bar{\theta}} (1 - F(\theta)) F(\theta)\, d\theta.\end{aligned}$$

With the uniform distribution, $F(\theta) = \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$ and $1 - F(\theta) = \frac{\bar{\theta} - \theta}{\bar{\theta} - \underline{\theta}}$. Thus,

$$\begin{aligned}\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha} \pi^*(\hat{\theta}_1, \hat{\theta}_2) &= (\hat{\theta}_2 - \underline{\theta}) \left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] + 2\int_{\hat{\theta}_2}^{\bar{\theta}} \left(-\theta^2 + \theta\left(\bar{\theta} + \underline{\theta}\right) - \bar{\theta}\underline{\theta}\right) d\theta \\ &= (\hat{\theta}_2 - \underline{\theta}) \left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] \\ &\quad + 2\left[\frac{1}{3}\hat{\theta}_2^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)(\bar{\theta}^2 - \hat{\theta}_2^2) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2)\right]. \end{aligned} \tag{25}$$

We next calculate $\Delta\pi^* \equiv \pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) - \pi^*(\hat{\theta}_1, \hat{\theta}_2)$:

$$\begin{aligned}\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha} \Delta\pi^* &= (\hat{\theta}_2 - \underline{\theta} + \epsilon) \left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon) - \frac{1}{2}\left((\hat{\theta}_2 + \epsilon)^2 - (\hat{\theta}_1 - \epsilon)^2\right)\right] \\ &\quad + 2\left[\frac{1}{3}(\hat{\theta}_2 + \epsilon)^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)\left(\bar{\theta}^2 - (\hat{\theta}_2 + \epsilon)^2\right) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2 - \epsilon)\right] \\ &\quad - (\hat{\theta}_2 - \underline{\theta}) \left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] \\ &\quad - 2\left[\frac{1}{3}\hat{\theta}_2^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)(\bar{\theta}^2 - \hat{\theta}_2^2) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2)\right]. \end{aligned} \tag{26}$$

On the right-hand side of (26), the terms proportional to ϵ sum up to

$$\begin{aligned}&\left[\left(\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}\hat{\theta}_2^2 + \frac{1}{2}\hat{\theta}_1^2\right) + (\hat{\theta}_2 - \underline{\theta})\left(2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1\right) + 2\left(\hat{\theta}_2^2 - \left(\bar{\theta} + \underline{\theta}\right)\hat{\theta}_2 + \bar{\theta}\underline{\theta}\right)\right]\epsilon \\ &\quad = (\hat{\theta}_2 - \hat{\theta}_1)\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)\right)\epsilon,\end{aligned}$$

and the terms proportional to ϵ^2 sum up to

$$\left[2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1 + 2\hat{\theta}_2 - \left(\bar{\theta} + \underline{\theta}\right)\right]\epsilon^2 = (\bar{\theta} - \underline{\theta} + \hat{\theta}_2 - \hat{\theta}_1)\epsilon^2.$$

Including all terms yields

$$\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha} \Delta\pi^* = (\hat{\theta}_2 - \hat{\theta}_1)\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)\right)\epsilon + (\bar{\theta} - \underline{\theta} + \hat{\theta}_2 - \hat{\theta}_1)\epsilon^2 + \frac{2}{3}\epsilon^3.$$

Because the right-hand side of the above is strictly positive for $\epsilon > 0$, and $1 - \alpha > 0$, we have $\Delta\pi^* > 0$, implying that spreading the cutoffs raises bidders' payoffs.

Together with $\Delta\Pi^*$ in (24), we have

$$\begin{aligned}
\left(\bar{\theta} - \underline{\theta}\right)^2 \left(\Delta\Pi^* - \Delta\pi^*\right) = & -\frac{2}{3}\epsilon^3 + \left(\hat{\theta}_1 - \hat{\theta}_2 + \underline{\theta}\right)\epsilon^2 + \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left(\underline{\theta} - \frac{1}{2}\left(\hat{\theta}_2 - \hat{\theta}_1\right)\right)\epsilon \\
& - (1-\alpha)\left\{(\hat{\theta}_2 - \hat{\theta}_1)\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)\right)\epsilon + (\bar{\theta} - \underline{\theta} + \hat{\theta}_2 - \hat{\theta}_1)\epsilon^2 + \frac{2}{3}\epsilon^3\right\} \\
= & \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left[2\underline{\theta} - \bar{\theta} - \left(\hat{\theta}_2 - \hat{\theta}_1\right) + \alpha\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}\left(\hat{\theta}_2 - \hat{\theta}_1\right)\right)\right]\epsilon \\
& + \left[(2-\alpha)\left(\underline{\theta} - \hat{\theta}_2 + \hat{\theta}_1\right) - (1-\alpha)\bar{\theta}\right]\epsilon^2 + o\left(\epsilon^3\right),
\end{aligned} \tag{27}$$

which is positive for small ϵ at $\hat{\theta}_2 = \hat{\theta}_1$, for all $\alpha \geq 0$, given the assumption that $2\underline{\theta} > \bar{\theta}$. $\Delta\Pi^* - \Delta\pi^*$ represents the increase in the seller's profit associated with adopting the more asymmetric mechanism, implying that interior symmetric cutoffs cannot be optimal.

Next, it follows from (21) and (25) that

$$\begin{aligned}
& \left(\bar{\theta} - \underline{\theta}\right)^2 \frac{\partial}{\partial\hat{\theta}_2}\left(\Pi^*(\hat{\theta}_1, \hat{\theta}_2) - \pi^*(\hat{\theta}_1, \hat{\theta}_2)\right)\Big|_{\hat{\theta}_1 = \underline{\theta}} \\
& = -\left(\frac{1}{2}\bar{\theta}^2 - \hat{\theta}_2\underline{\theta} + \frac{1}{2}\hat{\theta}_2^2\right) + \frac{1}{2}\bar{\theta}^2 - \frac{1}{2}\underline{\theta}^2 + \left(\bar{\theta} - \underline{\theta}\right)\phi \\
& \quad - (1-\alpha)\left\{(\hat{\theta}_2 - \underline{\theta})\left[\bar{\theta} - \frac{1}{2}(\hat{\theta}_2 + \underline{\theta})\right] + (\hat{\theta}_2 - \underline{\theta})(\bar{\theta} - \hat{\theta}_2) + 2\hat{\theta}_2^2 - 2\left(\bar{\theta} + \underline{\theta}\right)\hat{\theta}_2 + 2\bar{\theta}\underline{\theta}\right\} \\
& = -\frac{1}{2}(2-\alpha)(\hat{\theta}_2 - \underline{\theta})^2 + \left(\bar{\theta} - \underline{\theta}\right)\phi.
\end{aligned} \tag{28}$$

The right-hand side of (28) is strictly positive at $\hat{\theta}_2 = \underline{\theta}$, implying that $\hat{\theta}_2 = \hat{\theta}_1 = \underline{\theta}$ cannot be optimal, and it is immediate that $\hat{\theta}_2 = \hat{\theta}_1 = \bar{\theta}$ cannot be optimal. Thus, no symmetric mechanism is optimal.

Next, to show that always excluding a bidder is not optimal, suppose by way of contradiction that it is. Then under $2\underline{\theta} - \bar{\theta} > \phi$, setting $\hat{\theta}_1 = \underline{\theta}$ is optimal (given the premise that bidder 1 is the sole entrant). Differentiating (25) at $(\hat{\theta}_1 = \underline{\theta}, \hat{\theta}_2 = \bar{\theta})$ with respect to $\hat{\theta}_2$ yields

$$\begin{aligned}\frac{(\bar{\theta}-\underline{\theta})^2}{1-\alpha}\frac{\partial \pi^*(\hat{\theta}_1,\hat{\theta}_2)}{\partial \hat{\theta}_2}\bigg|_{\hat{\theta}_1\,=\underline{\theta},\,\hat{\theta}_2=\bar{\theta}} &= \bar{\theta}(\bar{\theta}-\underline{\theta}) - \frac{1}{2}(\bar{\theta}^2-\underline{\theta}^2) + (\bar{\theta}-\underline{\theta})\bar{\theta} - \bar{\theta}(\bar{\theta}-\underline{\theta}) \\ &\quad + 2(\bar{\theta}^2 - \bar{\theta}(\bar{\theta}+\underline{\theta}) + \bar{\theta}\underline{\theta}) \\ &= \frac{(\bar{\theta}-\underline{\theta})^2}{2} > 0.\end{aligned}$$

Moreover, by (22), $\frac{\partial \Pi^*(\hat{\theta}_1,\hat{\theta}_2)}{\partial \hat{\theta}_2}\Big|_{\hat{\theta}_1=\underline{\theta},\,\hat{\theta}_2=\bar{\theta}} < 0$, implying that

$$\frac{\partial}{\partial \hat{\theta}_2}\left(\Pi^*(\hat{\theta}_1,\hat{\theta}_2) - \pi^*(\hat{\theta}_1,\hat{\theta}_2)\right)\bigg|_{\hat{\theta}_1=\underline{\theta},\,\hat{\theta}_2=\bar{\theta}} < 0.$$

Thus, always excluding one bidder is not optimal; that is, $\hat{\theta}_2 \neq \bar{\theta}$. This completes the proof of part (i) of the proposition.

Because symmetric cutoffs are not optimal, assume that $\hat{\theta}_2 > \hat{\theta}_1$. To establish when $\hat{\theta}_1 = \underline{\theta}$ is optimal, observe that the leading term in (27) is ϵ, which has coefficient

$$\begin{aligned}&\left(\hat{\theta}_2-\hat{\theta}_1\right)\left[2\underline{\theta}-\bar{\theta}-\left(\hat{\theta}_2-\hat{\theta}_1\right)+\alpha\left(\bar{\theta}-\underline{\theta}+\frac{1}{2}\left(\hat{\theta}_2-\hat{\theta}_1\right)\right)\right] \\ &\geq \left(\hat{\theta}_2-\hat{\theta}_1\right)\left[2\underline{\theta}-\bar{\theta}-\left(\bar{\theta}-\underline{\theta}\right)+\alpha\left(\bar{\theta}-\underline{\theta}+\frac{1}{2}\left(\bar{\theta}-\underline{\theta}\right)\right)\right] \\ &= \left(\hat{\theta}_2-\hat{\theta}_1\right)\left[3\underline{\theta}-2\bar{\theta}+\frac{3}{2}\alpha\left(\bar{\theta}-\underline{\theta}\right)\right].\end{aligned}$$

This leading coefficient is positive (meaning that it is optimal to spread entry cutoffs until either $\hat{\theta}_1 = \underline{\theta}$ or $\hat{\theta}_2 = \bar{\theta}$) if and only if $\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$. But, because $\hat{\theta}_2 \neq \bar{\theta}$, we must have $\hat{\theta}_1 = \underline{\theta}$. Then, setting the right-hand side of the first-order condition (28) for $\hat{\theta}_2$ to zero yields

$$\hat{\theta}_2 = \underline{\theta} + \sqrt{\frac{2\phi\left(\bar{\theta}-\underline{\theta}\right)}{2-\alpha}} < \underline{\theta} + \sqrt{2\phi\left(\bar{\theta}-\underline{\theta}\right)} < \bar{\theta},$$

where the last inequality holds by $\bar{\theta}-\underline{\theta} > 2\phi$. Therefore, the optimal cutoffs are $\hat{\theta}_1 = \underline{\theta}$ and $\hat{\theta}_2 \in (\underline{\theta}, \bar{\theta})$, establishing part (ii) of the proposition. $\Box$

Proof of Proposition 4: We incorporate the cost of implementing asymmetric mechanisms to the two mechanisms considered in the proof of Proposition 3 (where in mechanism 2, $\hat{\theta}_1$ is replaced by $\hat{\theta}_1 - \epsilon$ and $\hat{\theta}_2$ replaced by $\hat{\theta}_2 + \epsilon$). The difference of this cost in the two mechanisms is $\gamma(\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon)^2 - \gamma(\hat{\theta}_2 - \hat{\theta}_1)^2 = 4\gamma[(\hat{\theta}_2 - \hat{\theta}_1)\epsilon + \epsilon^2]$. Subtracting this cost (multiplied by $(\bar{\theta} - \underline{\theta})^2$) from the right-hand side of (27) yields that

$$(\bar{\theta} - \underline{\theta})^2(\Delta\Pi^* - \Delta\pi^*) = (\hat{\theta}_2 - \hat{\theta}_1)\left[2\underline{\theta} - \bar{\theta} - (\hat{\theta}_2 - \hat{\theta}_1) + \alpha(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)) - 4(\bar{\theta} - \underline{\theta})^2\gamma\right]\epsilon$$
$$+ [(2-\alpha)(\underline{\theta} - \hat{\theta}_2 + \hat{\theta}_1) - (1-\alpha)\bar{\theta} - 4(\bar{\theta} - \underline{\theta})^2\gamma]\epsilon^2 + o(\epsilon^3), \tag{29}$$

where $\Delta\Pi^* - \Delta\pi^*$ is the increase in the seller's profit from adopting the more asymmetric mechanism derived in the proof of Proposition 3. For any symmetric mechanism with $\hat{\theta}_2 = \hat{\theta}_1 \in (\underline{\theta}, \bar{\theta})$, the right-hand side of (29) reduces to

$$[2\underline{\theta} - \bar{\theta} + \alpha(\bar{\theta} - \underline{\theta}) - 4(\bar{\theta} - \underline{\theta})^2\gamma]\epsilon^2 + o\left(\epsilon^3\right).$$

If $\gamma < \gamma^*(\alpha)$, this expression is strictly positive. Furthermore, $\hat{\theta}_2 = \hat{\theta}_1 = \underline{\theta}$ or $\hat{\theta}_2 = \hat{\theta}_1 = \bar{\theta}$ is not optimal. Thus, symmetric mechanisms are not optimal.

To show that when $\gamma \geq \gamma^*(\alpha)$, symmetric mechanisms are optimal, suppose instead that the optimal mechanism features $\hat{\theta}_2 > \hat{\theta}_1$. The leading term on the right-hand side of (29) is

$$\begin{aligned}
&(\hat{\theta}_2 - \hat{\theta}_1)\left(2\underline{\theta} - \bar{\theta} - (\hat{\theta}_2 - \hat{\theta}_1) + \alpha(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)) - 4(\bar{\theta} - \underline{\theta})^2\gamma\right)\epsilon \\
&\leq (\hat{\theta}_2 - \hat{\theta}_1)\left(2\underline{\theta} - \bar{\theta} - (\hat{\theta}_2 - \hat{\theta}_1) + \alpha(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)) - 4(\bar{\theta} - \underline{\theta})^2\gamma^*(\alpha)\right)\epsilon \\
&< (\hat{\theta}_2 - \hat{\theta}_1)\left(2\underline{\theta} - \bar{\theta} + \alpha(\bar{\theta} - \underline{\theta}) - 4(\bar{\theta} - \underline{\theta})^2\gamma^*(\alpha)\right)\epsilon \\
&= (\hat{\theta}_2 - \hat{\theta}_1)\left(2\underline{\theta} - \bar{\theta} + \alpha(\bar{\theta} - \underline{\theta}) - (2\underline{\theta} - \bar{\theta} + (\bar{\theta} - \underline{\theta})\alpha)\right)\epsilon = 0.
\end{aligned}$$

This implies that the seller would strictly benefit by slightly closing the gaps between $\hat{\theta}_2$ and $\hat{\theta}_1$ (i.e., $\epsilon < 0$). Thus, $\hat{\theta}_2 > \hat{\theta}_1$ cannot be optimal, implying that symmetric mechanisms are optimal if $\gamma \geq \gamma^*(\alpha)$. $\square$

Proof of Proposition 5: By (11), in any optimal mechanism with entry cutoffs $(\hat{\theta}_1, \hat{\theta}_2)$, the seller's revenue is

$$\Pi^*(\hat{\theta}_1, \hat{\theta}_2) = \int_{\hat{\theta}_2}^{\bar{\theta}}\int_{\hat{\theta}_1}^{\bar{\theta}} \max\{\xi(\theta_1), \xi(\theta_2)\}\, dF(\theta_1)\, dF(\theta_2)$$
$$+ F(\hat{\theta}_2)\int_{\hat{\theta}_1}^{\bar{\theta}} \xi(\theta_1) dF(\theta_1) + F(\hat{\theta}_1)\int_{\hat{\theta}_2}^{\bar{\theta}} \xi(\theta_2) dF(\theta_2) - \sum_i \phi_i(1 - F(\hat{\theta}_i)).$$

We first show that inducing $\hat{\theta}_1 > \hat{\theta}_2$ cannot be optimal. Suppose it is. All terms in the seller's revenue save for the last one are symmetric in $\hat{\theta}_1$ and $\hat{\theta}_2$. Therefore, interchanging $\hat{\theta}_1$ and $\hat{\theta}_2$ would increase the seller's revenue because $\phi_2 > \phi_1$. It follows that $\hat{\theta}_1 \leq \hat{\theta}_2$.

For $\hat{\theta}_1 \leq \hat{\theta}_2$, the seller's revenue can be rewritten as

$$\Pi^*(\hat{\theta}_1, \hat{\theta}_2) = \int_{\hat{\theta}_2}^{\bar{\theta}} \left(\int_{\hat{\theta}_1}^{\theta_2} \xi(\theta_2) dF(\theta_1) + \int_{\theta_2}^{\bar{\theta}} \xi(\theta_1) dF(\theta_1) \right) dF(\theta_2)$$
$$+ F(\hat{\theta}_2) \int_{\hat{\theta}_1}^{\bar{\theta}} \xi(\theta_1) dF(\theta_1) + F(\hat{\theta}_1) \int_{\hat{\theta}_2}^{\bar{\theta}} \xi(\theta_2) dF(\theta_2) - \sum_i \phi_i (1 - F(\hat{\theta}_i)).$$

Suppose $\hat{\theta}_2 = \bar{\theta}$. Then, because $2\underline{\theta} - \bar{\theta} > \phi_1$ implies $\xi(\underline{\theta}) > \phi_1$, $\hat{\theta}_1 = \underline{\theta}$ is optimal. However,

$$\begin{aligned}
\left.\frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_2}\right|_{\hat{\theta}_1 = \underline{\theta},\, \hat{\theta}_2 = \bar{\theta}} &= f(\bar{\theta}) \left(\int_{\underline{\theta}}^{\bar{\theta}} (\xi(\theta_1) - \xi(\bar{\theta})) dF(\theta_1) + \phi_2 \right) \\
&= f(\bar{\theta}) \left(\int_{\underline{\theta}}^{\bar{\theta}} ((2-\alpha)\theta_1 - (2-\alpha)\bar{\theta}) dF(\theta_1) + \phi_2 \right) \\
&= f(\bar{\theta}) \left(-(2-\alpha) \frac{\bar{\theta} - \underline{\theta}}{2} + \phi_2 \right) \\
&< f(\bar{\theta}) \left(-\frac{\bar{\theta} - \underline{\theta}}{2} + \phi_2 \right) < 0,
\end{aligned}$$

where the last inequality holds by $\frac{\bar{\theta} - \underline{\theta}}{2} > \phi_2$. Thus, $\hat{\theta}_2 < \bar{\theta}$.

Next consider any symmetric entry cutoff $\hat{\theta}' = \hat{\theta}_1 = \hat{\theta}_2 \in (\underline{\theta}, \bar{\theta})$. Then,

$$\begin{aligned}
&\left.\left(\frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_2} - \frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_1} \right)\right|_{\hat{\theta}' = \hat{\theta}_1 = \hat{\theta}_2} \\
&= -f(\hat{\theta}') \int_{\hat{\theta}'}^{\bar{\theta}} \xi(\theta) dF(\theta) + f(\hat{\theta}') \int_{\hat{\theta}'}^{\bar{\theta}} \xi(\theta) dF(\theta) - F(\hat{\theta}') \xi(\hat{\theta}') f(\hat{\theta}') + f(\hat{\theta}') \int_{\hat{\theta}'}^{\bar{\theta}} \xi(\theta) dF(\theta) \\
&\quad + F(\hat{\theta}') \xi(\hat{\theta}') f(\hat{\theta}') - f(\hat{\theta}') \int_{\hat{\theta}'}^{\bar{\theta}} \xi(\theta) dF(\theta) + (\phi_2 - \phi_1) f(\hat{\theta}') \\
&= (\phi_2 - \phi_1) f(\hat{\theta}') > 0.
\end{aligned}$$

The last equality holds by the full-support assumption and $\phi_2 > \phi_1$: reducing $\hat{\theta}_1$ and increasing $\hat{\theta}_2$ marginally from any symmetric cutoff $\hat{\theta}' \in (\underline{\theta}, \bar{\theta})$ increase seller revenues. Thus,

$\hat{\theta}_1 = \hat{\theta}_2 \in (\underline{\theta}, \bar{\theta})$ cannot be optimal. Moreover,

$$\left.\frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_2}\right|_{\hat{\theta}_1 = \hat{\theta}_2 = \underline{\theta}} = \phi_2 f(\underline{\theta}) > 0$$

implies that $\hat{\theta}_1 = \hat{\theta}_2 = \underline{\theta}$ is not optimal. Therefore, we must have $\hat{\theta}_1 < \hat{\theta}_2 < \bar{\theta}$. $\square$

Proof of Proposition 6: Let $(\hat{\theta}_1, \hat{\theta}_2)$ be entry cutoffs. Let $\Pi(\hat{\theta}_1, p)$ be the seller's expected revenue in the optimal mechanism, where $p \in [0,1]$ is the probability that bidder 2 enters.

With uniformly-distributed valuations, the virtual valuation of bidder $i \in \{1,2\}$ is given by (17). Thus $\lim_{\bar{\theta}_2 \to \underline{\theta}} \xi_2(\theta_2) = \underline{\theta}$ for all θ_2. Let $\tilde{\theta}$ be the value of θ_1 such that $\xi_1(\theta_1) = \underline{\theta}$:

$$\tilde{\theta} = \frac{\underline{\theta} + (1-\alpha)\,\bar{\theta}_1}{2-\alpha}.$$

To calculate $\Pi(\hat{\theta}_1, p)$, there are two cases to consider:

Case 1: $\hat{\theta}_1 \geq \tilde{\theta}$. Noting that with $\bar{\theta}_2 \to \underline{\theta}$, the (very) weak bidder 2 wins only when strong bidder 1 does not enter and bidder 2 enters (which occurs with probability p), (11) yields that

$$\begin{aligned}\Pi(\hat{\theta}_1 \geq \tilde{\theta}, p) &= \int_{\hat{\theta}_1}^{\bar{\theta}_1} \xi_1(\theta_1) dF_1(\theta_1) + F_1(\hat{\theta}_1) p\underline{\theta} - \Big((1 - F_1(\hat{\theta}_1)) + p\Big)\phi \\ &= \frac{1}{\bar{\theta}_1 - \underline{\theta}}\left(\left(\frac{\alpha}{2} - 1\right)\hat{\theta}_1^2 + (1-\alpha)\,\bar{\theta}_1\hat{\theta}_1 + \frac{\alpha}{2}\bar{\theta}_1^2 + (\hat{\theta}_1 - \underline{\theta})p\underline{\theta} - (\bar{\theta}_1 - \hat{\theta}_1 + p(\bar{\theta}_1 - \underline{\theta}))\phi\right).\end{aligned} \tag{30}$$

The first term is the contribution to Π when $\theta_1 \in [\hat{\theta}_1, \bar{\theta}_1]$, the second term is when $\theta_1 \in [\underline{\theta}, \hat{\theta}_1)$.

Case 2: $\hat{\theta}_1 < \tilde{\theta}$. Then, with $\bar{\theta}_2 \to \underline{\theta}$, bidder 1 always wins when $\theta_1 \in [\tilde{\theta}, \bar{\theta}_1]$; while bidder 2 wins if it enters (with probability p) and $\theta_1 \in [\hat{\theta}_1, \tilde{\theta})$. Thus, (11) yields

$$\begin{aligned}\Pi(\hat{\theta}_1 < \tilde{\theta}, p) &= \int_{\tilde{\theta}}^{\bar{\theta}_1} \xi_1(\theta_1) dF_1(\theta_1) + \int_{\hat{\theta}_1}^{\tilde{\theta}} (p\underline{\theta} + (1-p)\,\xi_1(\theta_1))\, dF_1(\theta_1) \\ &\quad + F_1(\hat{\theta}_1) p\underline{\theta} - \Big((1 - F_1(\hat{\theta}_1)) + p\Big)\phi \\ &= \frac{1}{\bar{\theta}_1 - \underline{\theta}}\left(\left(\frac{\alpha}{2} - 1\right)\tilde{\theta}^2 + (1-\alpha)\,\bar{\theta}_1\tilde{\theta} + \frac{\alpha}{2}\bar{\theta}_1^2\right) \\ &\quad + \frac{1}{\bar{\theta}_1 - \underline{\theta}}\left[p\underline{\theta}(\tilde{\theta} - \hat{\theta}_1) + (1-p)(\tilde{\theta} - \hat{\theta}_1)\left(\left(1 - \frac{\alpha}{2}\right)(\hat{\theta}_1 + \tilde{\theta}) - (1-\alpha)\,\bar{\theta}_1\right)\right] \\ &\quad + \frac{\hat{\theta}_1 - \underline{\theta}}{\bar{\theta}_1 - \underline{\theta}} p\underline{\theta} - \left(\frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} + p\right)\phi,\end{aligned}$$

where the first term is the contribution to Π when $\theta_1 \in [\tilde{\theta}, \bar{\theta}_1]$, the second term is when $\theta_1 \in [\hat{\theta}_1, \tilde{\theta})$, and the third term is when $\theta_1 \in [\underline{\theta}, \hat{\theta}_1)$.

The optimal choices of $\hat{\theta}_1$ and p maximize expected seller revenue:

$$(\hat{\theta}_1^*, p^*) \in \arg \max_{\hat{\theta}_1 \in [\underline{\theta}, \bar{\theta}_1],\, p \in [0,1]} \Pi(\hat{\theta}_1, p).$$

We denote the maximum value of Π by $\Pi^* \equiv \Pi(\hat{\theta}_1^*, p^*)$.

Lemma 4 *Let* $\Pi^*_{p=0} \equiv \max_{\hat{\theta}_1 \in [\underline{\theta}, \bar{\theta}_1]} \Pi(\hat{\theta}_1, p=0)$ *and* $\Pi^*_{p=1} \equiv \max_{\hat{\theta}_1 \in [\underline{\theta}, \bar{\theta}_1]} \Pi(\hat{\theta}_1, p=1)$. *Then,*

(i) $\Pi^* = \max \left\{ \Pi^*_{p=0}, \Pi^*_{p=1} \right\}$, *and*

(ii) If $\Pi^*_{p=0} > \Pi^*_{p=1}$, *then* $p^* = 0$ *in all solutions; if* $\Pi^*_{p=0} < \Pi^*_{p=1}$, *then* $p^* = 1$ *in all solutions; and if* $\Pi^*_{p=0} = \Pi^*_{p=1}$, *then solutions feature different* p^*, *and there exists at least one solution for each* $p^* \in [0,1]$.

Proof: Although the form of $\Pi(\hat{\theta}_1, p)$ is complicated, it has the simple feature that $\Pi(\hat{\theta}_1, p)$ is affine in p. In particular, regardless of whether $\hat{\theta}_1 \geq \tilde{\theta}$ or $\hat{\theta}_1 < \tilde{\theta}$, $\frac{\partial \Pi(\hat{\theta}_1, p)}{\partial p}$ is independent of p:

$$\frac{\partial \Pi(\hat{\theta}_1 \geq \tilde{\theta}, p)}{\partial p} = \frac{\hat{\theta}_1 - \underline{\theta}}{\bar{\theta}_1 - \underline{\theta}} \underline{\theta} - \phi;$$
$$\frac{\partial \Pi(\hat{\theta}_1 < \tilde{\theta}, p)}{\partial p} = \frac{\tilde{\theta} - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left(\underline{\theta} - \left(1 - \frac{\alpha}{2}\right) \left(\hat{\theta}_1 + \tilde{\theta}\right) + (1-\alpha)\,\bar{\theta}_1 \right) + \frac{\hat{\theta}_1 - \underline{\theta}}{\bar{\theta}_1 - \underline{\theta}} \underline{\theta} - \phi.$$

Utilizing this feature, we consider the value of $\frac{\partial \Pi(\hat{\theta}_1^*, p^*)}{\partial p}$, where $(\hat{\theta}_1^*, p^*)$ is a solution that maximizes $\Pi(\hat{\theta}_1, p)$. There can be only three cases:

1. If $\frac{\partial \Pi(\hat{\theta}_1^*, p)}{\partial p} < 0$, then $p^* = 0$. To see this, suppose to the contrary that $p^* > 0$. Then $\Pi(\hat{\theta}_1^*, p=0) > \Pi(\hat{\theta}_1^*, p^*)$, contradicting the premise that $(\hat{\theta}_1^*, p^*)$ is a solution for optimality. Thus $p^* = 0$. Furthermore, it follows that $\Pi^*_{p=0} > \Pi^*_{p=1}$.

2. If $\frac{\partial \Pi(\hat{\theta}_1^*, p^*)}{\partial p} > 0$, then by same logic, we have $p^* = 1$ and $\Pi^*_{p=0} < \Pi^*_{p=1}$.

3. If $\frac{\partial \Pi(\hat{\theta}_1^*, p^*)}{\partial p} = 0$, then for any $p \in [0,1]$, $(\hat{\theta}_1^*, p)$ is also optimal. $\square$

We now use this lemma to solve for all $(\hat{\theta}_1^*, p^*)$. We have

$$\frac{\partial \Pi(\hat{\theta}_1 \geq \tilde{\theta}, p)}{\partial \hat{\theta}_1} = \frac{1}{\bar{\theta}_1 - \underline{\theta}} \left((\alpha - 2)\hat{\theta}_1 + (1-\alpha)\bar{\theta}_1 + p\underline{\theta} + \phi \right); \tag{31}$$

$$\frac{\partial \Pi(\hat{\theta}_1 < \tilde{\theta}, p)}{\partial \hat{\theta}_1} = \frac{1}{\bar{\theta}_1 - \underline{\theta}} \left[(1-p) \left((\alpha - 2)\hat{\theta}_1 + (1-\alpha)\bar{\theta}_1 \right) + \phi \right]. \tag{32}$$

(Step 1): We first calculate $\Pi^*_{p=0}$. We show

$$\Pi^*_{p=0} = \begin{cases} \underline{\theta} + \frac{\alpha}{2}\left(\bar{\theta}_1 - \underline{\theta}\right) - \phi & \text{if} \quad \phi \leq \xi_1(\underline{\theta}) = (2-\alpha)\,\underline{\theta} - (1-\alpha)\,\bar{\theta}_1 \\ \frac{1}{4-2\alpha} \frac{\left(\bar{\theta}_1 - \phi\right)^2}{\bar{\theta}_1 - \underline{\theta}} & \text{if} \quad \phi > (2-\alpha)\,\underline{\theta} - (1-\alpha)\,\bar{\theta}_1. \end{cases} \tag{33}$$

To establish (33), noting that conditional on $p = 0$, bidder 1 is the only potential bidder, it follows from (11) that $\Pi(\hat{\theta}_1, p = 0)$ is maximized by (19), where the $\hat{\theta}_1$ in the second line of (19) solves the first-order condition $\xi_1(\hat{\theta}_1) = \phi$. By (11) and (17),

$$\begin{aligned} \Pi(\hat{\theta}_1, p = 0) &= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \left[\left(1 - \frac{\alpha}{2}\right) \left(\left(\bar{\theta}_1\right)^2 - \left(\hat{\theta}_1\right)^2 \right) - (1-\alpha)\,\bar{\theta}_1 \left(\bar{\theta}_1 - \hat{\theta}_1\right) \right] - \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}}\phi \\ &= \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left[\left(1 - \frac{\alpha}{2}\right) \left(\bar{\theta}_1 + \hat{\theta}_1\right) - (1-\alpha)\,\bar{\theta}_1 - \phi \right] \\ &= \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left[\frac{\alpha}{2}\bar{\theta}_1 + \left(1 - \frac{\alpha}{2}\right)\hat{\theta}_1 - \phi \right]. \end{aligned} \tag{34}$$

If $\phi \leq (2-\alpha)\,\underline{\theta} - (1-\alpha)\,\bar{\theta}_1$, then, plugging the first line of (19) ($\hat{\theta}_1 = \underline{\theta}$) into (34) establishes the first line of (33). If, instead, $\phi \in \left(\xi_1(\underline{\theta}), \bar{\theta}_1\right)$, then plugging the second line of (19) into (34) yields

$$\begin{aligned} \Pi^*_{p=0} &= \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left[\frac{\alpha}{2}\bar{\theta}_1 + \left(1 - \frac{\alpha}{2}\right) \frac{(1-\alpha)\,\bar{\theta}_1 + \phi}{(2-\alpha)} - \phi \right] \\ &= \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left[\frac{\alpha}{2}\bar{\theta}_1 + \frac{(1-\alpha)\,\bar{\theta}_1 + \phi}{2} - \phi \right] \\ &= \frac{\bar{\theta}_1 - \hat{\theta}_1}{\bar{\theta}_1 - \underline{\theta}} \left[\frac{1}{2}\bar{\theta}_1 - \frac{1}{2}\phi \right] \\ &= \frac{\left(\bar{\theta}_1 - \phi\right)}{\bar{\theta}_1 - \underline{\theta}} \frac{1}{2-\alpha} \left[\frac{1}{2}\bar{\theta}_1 - \frac{1}{2}\phi \right], \end{aligned}$$

which establishes the second line of (33).

(Step 2): We next calculate $\Pi^*_{p=1}$. We show

$$\Pi^*_{p=1} = \begin{cases} \underline{\theta} - \phi & \text{if} \quad \phi \geq \bar{\theta}_1 - \underline{\theta} \\ \hat{\Pi} & \text{if} \quad \phi < \bar{\theta}_1 - \underline{\theta}, \end{cases} \tag{35}$$

where

$$\hat{\Pi} \equiv \frac{1}{\bar{\theta}_1 - \underline{\theta}} \times \left\{ \frac{\big((1-\alpha)\,\bar{\theta}_1 + \underline{\theta} + \phi\big)^2}{2\,(2-\alpha)} + \frac{\alpha}{2}\bar{\theta}_1^2 - \underline{\theta}^2 - 2\bar{\theta}_1\phi + \underline{\theta}\phi \right\}. \tag{36}$$

To establish (35), note that (32) yields

$$\frac{\partial \Pi(\hat{\theta}_1 < \tilde{\theta}, p=1)}{\partial \hat{\theta}_1} = \frac{\phi}{\bar{\theta}_1 - \underline{\theta}} > 0,$$

by which we have $\Pi^*_{p=1} = \max_{\hat{\theta}_1 \in [\underline{\theta}, \bar{\theta}_1]} \Pi(\hat{\theta}_1, p=1) = \max_{\hat{\theta}_1 \geq \tilde{\theta}} \Pi(\hat{\theta}_1, p=1)$. Then (31) yields

$$\frac{\partial \Pi(\hat{\theta}_1 \geq \tilde{\theta}, p=1)}{\partial \hat{\theta}_1} = \frac{1}{\bar{\theta}_1 - \underline{\theta}} \Big((\alpha-2)\hat{\theta}_1 + (1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi \Big),$$

which is decreasing in $\hat{\theta}_1$. Moreover, evaluating it both at $\tilde{\theta}$ and $\bar{\theta}_1$ yields that

$$\begin{aligned} \frac{\partial \Pi(\hat{\theta}_1 = \tilde{\theta}, p=1)}{\partial \hat{\theta}_1} &= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \Big((\alpha-2)\tilde{\theta} + (1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi \Big) \\ &= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \Big(-\xi_1(\tilde{\theta}) + \underline{\theta} + \phi \Big) = \frac{\phi}{\bar{\theta}_1 - \underline{\theta}} > 0; \\ \frac{\partial \Pi(\hat{\theta}_1 = \bar{\theta}_1, p=1)}{\partial \hat{\theta}_1} &= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \left(-\bar{\theta}_1 + \underline{\theta} + \phi \right). \end{aligned}$$

Thus, $\frac{\partial \Pi(\hat{\theta}_1 = \bar{\theta}_1, p=1)}{\partial \hat{\theta}_1} \geq 0$ if and only if $\phi \geq \bar{\theta}_1 - \underline{\theta}$. Hence, if $\phi \geq \bar{\theta}_1 - \underline{\theta}$, then $\hat{\theta}_1 = \bar{\theta}_1$ maximizes $\Pi(\hat{\theta}_1, p=1)$; evaluating (30) at $\hat{\theta}_1 = \bar{\theta}_1$ and $p=1$ yields the first line of (35).

Now consider $\phi < \bar{\theta}_1 - \underline{\theta}$. Because $\frac{\partial \Pi(\hat{\theta}_1 \geq \tilde{\theta}, p=1)}{\partial \hat{\theta}_1}$ declines with $\hat{\theta}_1$, the first-order condition $\frac{\partial \Pi(\hat{\theta}_1 \geq \tilde{\theta}, p=1)}{\partial \hat{\theta}_1} = 0$ is necessary and sufficient for maximization. Plugging the solution for the first-order condition, (18), and $p=1$ into (30) yields

$$\begin{aligned} \Pi^*_{p=1} = \frac{1}{\bar{\theta}_1 - \underline{\theta}} \times \Bigg\{ & \left(\frac{\alpha}{2} - 1\right) \left(\frac{(1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi}{2-\alpha} \right)^2 + (1-\alpha)\,\bar{\theta}_1 \frac{(1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi}{2-\alpha} \\ & + \frac{\alpha}{2}\bar{\theta}_1^2 + \left(\frac{(1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi}{2-\alpha} - \underline{\theta} \right) \underline{\theta} - \left(2\bar{\theta}_1 - \frac{(1-\alpha)\bar{\theta}_1 + \underline{\theta} + \phi}{2-\alpha} - \underline{\theta} \right) \phi \Bigg\}, \end{aligned}$$

yielding the second line of (35).

Finally, consider the difference $\Pi^*_{p=0} - \Pi^*_{p=1}$. Optimality of $\Pi^*_{p=0}$ yields $\Pi^*_{p=0} \geq \Pi(\hat{\theta}_1 =$

$\underline{\theta}, p = 0) = \underline{\theta} + \frac{\alpha}{2}\left(\bar{\theta}_1 - \underline{\theta}\right) - \phi$ for all ϕ. By (35), $\Pi^*_{p=1} = \underline{\theta} - \phi$ if $\phi \geq \bar{\theta}_1 - \underline{\theta}$. Thus, $\Pi^*_{p=0} \geq \Pi^*_{p=1}$ if $\phi \geq \bar{\theta}_1 - \underline{\theta}$, and strict inequality holds if $\alpha > 0$.

Under $\phi < \bar{\theta}_1 - \underline{\theta}$, we first show $\Pi^*_{p=0} < \Pi^*_{p=1}$ when ϕ approaches 0. By (33) and (35),

$$\begin{aligned}
\Pi^*_{p=1} - \Pi^*_{p=0} &= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \times \left\{ \frac{\left((1-\alpha)\,\bar{\theta}_1 + \underline{\theta}\right)^2}{2\,(2-\alpha)} + \frac{\alpha}{2}\bar{\theta}_1^2 - \underline{\theta}^2 \right\} - \underline{\theta} - \frac{\alpha}{2}\left(\bar{\theta}_1 - \underline{\theta}\right) \\
&= \frac{1}{\bar{\theta}_1 - \underline{\theta}} \times \left\{ \frac{\left((1-\alpha)\bar{\theta}_1 + \underline{\theta}\right)^2}{2(2-\alpha)} - \left((1-\alpha)\bar{\theta}_1 + \frac{\alpha}{2}\underline{\theta}\right)\underline{\theta} \right\} \\
&= \frac{1}{(\bar{\theta}_1 - \underline{\theta})2(2-\alpha)} \times \left\{ \left((1-\alpha)\bar{\theta}_1 + \underline{\theta}\right)^2 - 2(2-\alpha)\left((1-\alpha)\bar{\theta}_1 + \frac{\alpha}{2}\underline{\theta}\right)\underline{\theta} \right\} \\
&= \frac{1}{(\bar{\theta}_1 - \underline{\theta})2(2-\alpha)} \times \left\{ (1-\alpha)^2\bar{\theta}_1^2 - 2(1-\alpha)^2\bar{\theta}_1\underline{\theta} + (1-\alpha)^2\underline{\theta}^2 \right\} \\
&= \frac{(1-\alpha)^2(\bar{\theta}_1 - \underline{\theta})}{2(2-\alpha)} > 0.
\end{aligned}$$

Moreover, differentiating (33) with respect to ϕ yields

$$\frac{\partial \Pi^*_{p=0}}{\partial \phi} \begin{cases} = -1 & \text{if} \quad \phi \leq (2-\alpha)\,\underline{\theta} - (1-\alpha)\,\bar{\theta}_1 \\ \geq -1 & \text{if} \quad \phi > (2-\alpha)\,\underline{\theta} - (1-\alpha)\,\bar{\theta}_1, \end{cases}$$

and, for $\phi \in \left(0, \bar{\theta}_1 - \underline{\theta}\right)$, differentiating the second line of (35) with respect to ϕ yields

$$\frac{\partial \Pi^*_{p=1}}{\partial \phi} = -1 + \frac{\phi - (\bar{\theta}_1 - \underline{\theta})}{(2-\alpha)(\bar{\theta}_1 - \underline{\theta})} < -1.$$

Hence, $\Pi^*_{p=0} - \Pi^*_{p=1}$ strictly increases in ϕ for $\phi \in (0, \bar{\theta}_1 - \underline{\theta})$. When $\alpha > 0$, because (1) $\Pi^*_{p=0} - \Pi^*_{p=1} < 0$ when ϕ approaches zero, (2) $\Pi^*_{p=0} - \Pi^*_{p=1} > 0$ when $\phi \geq \bar{\theta}_1 - \underline{\theta}$, and (3) $\Pi^*_{p=0} - \Pi^*_{p=1}$ is continuous in ϕ (see (33) and (35)), there exists a unique $\phi^*(\alpha) < \bar{\theta}_1 - \underline{\theta}$ such that $\Pi^*_{p=0} - \Pi^*_{p=1} = 0$, or

$$\Pi^*_{p=0}\left(\alpha, \phi^*\right) - \hat{\Pi}\left(\alpha, \phi^*\right) = 0, \tag{37}$$

where $\hat{\Pi}$ is given by (36). This $\phi^*(\alpha)$ has the property that $\Pi^*_{p=0} < \Pi^*_{p=1}$ for all $\phi < \phi^*(\alpha)$ and $\Pi^*_{p=0} > \Pi^*_{p=1}$ for all $\phi \in \left(\phi^*(\alpha), \bar{\theta}_1\right)$.

To determine the sign of $\frac{d\phi^*(\alpha)}{d\alpha}$, we apply the implicit function theorem to (37) to obtain

$$\frac{d\phi^*(\alpha)}{d\alpha} = -\frac{\frac{d\left(\Pi^*_{p=0} - \hat{\Pi}\right)}{d\alpha}}{\frac{d\left(\Pi^*_{p=0} - \hat{\Pi}\right)}{d\phi^*(\alpha)}}.$$

By (36), we have

$$\begin{aligned}
\frac{d}{d\alpha}\hat{\Pi} &\equiv \frac{1}{\bar{\theta}_1-\underline{\theta}} \times \left\{ -\frac{\bar{\theta}_1\left((1-\alpha)\bar{\theta}_1+\underline{\theta}+\phi\right)}{2-\alpha} + \frac{\left((1-\alpha)\bar{\theta}_1+\underline{\theta}+\phi\right)^2}{2(2-\alpha)^2} + \frac{1}{2}\bar{\theta}_1^2 \right\} \\
&= \frac{1}{\bar{\theta}_1-\underline{\theta}}\frac{1}{2(2-\alpha)^2}\left\{ \left((1-\alpha)\bar{\theta}_1+\underline{\theta}+\phi\right)^2 - \bar{\theta}_1\left((1-\alpha)\bar{\theta}_1+\underline{\theta}+\phi\right)2(2-\alpha) + (2-\alpha)^2\bar{\theta}_1^2 \right\} \\
&= \frac{1}{\bar{\theta}_1-\underline{\theta}}\frac{\left(\bar{\theta}_1-\underline{\theta}-\phi\right)^2}{2(2-\alpha)^2},
\end{aligned}$$

and by (33) we have

$$\frac{d}{d\alpha}\Pi^*_{p=0} = \begin{cases} \frac{1}{2}\left(\bar{\theta}_1-\underline{\theta}\right) & \text{if } \phi \leq (2-\alpha)\underline{\theta} - (1-\alpha)\bar{\theta}_1 \\ \frac{1}{2(2-\alpha)^2}\frac{\left(\bar{\theta}_1-\phi\right)^2}{\bar{\theta}_1-\underline{\theta}} & \text{if } \phi > (2-\alpha)\underline{\theta} - (1-\alpha)\bar{\theta}_1. \end{cases}$$

Noting that $2-\alpha>1$ and $\bar{\theta}_1-\underline{\theta}-\phi<\bar{\theta}_1-\underline{\theta}$, we readily have $\frac{d\left(\Pi^*_{p=0}-\hat{\Pi}\right)}{d\alpha}>0$. Because $\frac{d\left(\Pi^*_{p=0}-\hat{\Pi}\right)}{d\phi^*(\alpha)}>0$ for $\phi^*(\alpha)<\bar{\theta}_1-\underline{\theta}$, we have $\frac{d\phi^*(\alpha)}{d\alpha}<0$.

Finally, note that $\lim_{\phi\to\phi^*(\alpha)^-}\hat{\theta}_1$ is given by (18) evaluated at $\phi=\phi^*(\alpha)$, and $\lim_{\phi\to\phi^*(\alpha)^+}\hat{\theta}_1$ is given by (19) evaluated at $\phi=\phi^*(\alpha)$. Thus, $\lim_{\phi\to\phi^*(\alpha)^-}\hat{\theta}_1>\lim_{\phi\to\phi^*(\alpha)^+}\hat{\theta}_1$. Furthermore, $\hat{\theta}_1$ in (19) weakly increases in ϕ. This completes the proof. □

Unknown Valuations. Suppose that bidders do not know their valuations before making entry decisions. Then, if m potential bidders enter, defining $Q_m(\theta^1)$ to be the distribution over the highest valuation, a seller's expected payoff cannot exceed

$$\bar{\Pi}_m \equiv \int_{\underline{\theta}}^{\bar{\theta}} \theta^1 dQ_m\left(\theta^1\right) - m\phi.$$

This reflects that social welfare cannot exceed $\int_{\underline{\theta}}^{\bar{\theta}}\theta^1 dQ_m\left(\theta^1\right)$, and expected bidder payoffs (net of entry costs) must be nonnegative. Thus, an upper bound on a seller's expected payoff is $\bar{\Pi}_{m^*}$, where $m^*=\arg\max_{m\leq n}\bar{\Pi}_m$. It follows that $m^*\geq 1$ when $\phi<\underline{\theta}$. We now show that $\bar{\Pi}_{m^*}$ is attainable simply by using lump-sum transfers (the same fee for each bidder, where a negative entry fee corresponds to a cash reimbursement):

Proposition 7 $\bar{\Pi}_{m^*}$ *is implementable in the pure-strategy equilibrium of any standard format in which bidders bid with a fixed royalty rate* $\alpha\in[0,1)$ *plus cash, face a reserve that does not exceed the break-even bid of a bidder with valuation* $\underline{\theta}$*, and pay an entry fee of* $\pi^*-\phi$*, where*

π^* is the expected payoff of an entering bidder (excluding entry costs) given m^* entrants.

Proof: We show that m^* potential bidders' entering constitutes an equilibrium. Any entering bidder receives expected payoff (gross of entry cost) of ϕ. Thus, entering is a best response. Further, if $n > m^*$, then each potential bidder who did not enter strictly prefers not to enter: the expected payoff (gross of entry cost) from entry would be strictly less than ϕ due to the heightened competition, making entering unprofitable. Thus, the equilibrium holds. In equilibrium, each bidder's ex-ante expected payoff (including entry costs) is zero, and social welfare is maximized for the given m^* entrants, establishing the proposition. $\square$

One way to implement this mechanism is to use $\alpha = 0$, i.e., pure cash auctions (hence, no tying) and an entry fee.[19] By contrast, if bidders know their valuations before entry, potential bidders have an informational advantage that a seller must offset by tying payments to their private information, as in Lemmas 1 and 2. Further, with unknown-valuations, efficiency is not impaired by having no trade—a seller always awards the asset, as the profit equals the welfare gain from trade. In contrast, with known valuations, a seller raises entry thresholds, screening out low-valuation bidders.

When bidders do not know their valuations prior to making entry decisions, two types of equilibria exist: a pure strategy equilibrium (McAfee and McMillan, 1987) in which entrants expect non-negative profits, but with greater entry, expected profits would become negative; and a mixed strategy equilibrium (Levin and Smith, 1994) in which potential bidders enter with a common probability p. The equilibrium in Proposition 7 delivers the optimal number of entrants: full surplus extraction is obtained via the pure-strategy equilibrium, as in McAfee and McMillan (1987), in which the right (deterministic) number of bidders endogenously choose to enter, making it unnecessary to restrict entry.

References

[1] Andrade, Gregor, Mark Mitchell, and Erik Stafford. 2001. "New Evidence and Perspectives on Mergers." *Journal of Economic Perspectives* 15(2): 103–120.

[19] Alternatively, if there exists $\alpha \in [0, 1)$ such that $\pi^* = \phi$, then the mechanism in Proposition 7 can be implemented without charging an entry fee.

[2] Athey, Susan, Dominic Coey, and Jonathan Levin. 2013. "Set-asides and subsidies in auctions." *American Economic Journal: Microeconomics* 5(1): 1–27.

[3] Ayres, Ian, and Peter Cramton. 1996. "Deficit Reduction Through Diversity: How Affirmative Action at the FCC Increased Auction Competition." *Stanford Law Review* 48(4): 761–815.

[4] Bates, Tom and Michael Lemmon. 2003. "Breaking up is hard to do? An analysis of termination fee provisions and merger outcomes." *Journal of Financial Economics* 69(3) : 469–504.

[5] Bhattacharya, Vivek, James Roberts, and Andrew Sweeting. 2014. "Regulating bidder participation in auctions." *RAND Journal of Economics* 45(4): 675–704.

[6] Campbell, C. (1998), "Coordination in Auctions with Entry." *Journal of Economic Theory* 82(2), 425–450.

[7] Celik, Gorkem and Okan Yilankaya. 2009. "Optimal Auctions with Simultaneous and Costly Participation." *The B.E. Journal of Theoretical Economics* 9(1): 1935–1704

[8] Cremer, Jacques. 1987. "Auctions with Contingent Payments: Comment." *American Economic Review* 77(4): 746.

[9] Deb, Rajul, and Mallesh Pai. 2017. "Discrimination via Symmetric Auctions." *American Economic Journal: Microeconomics* 9(1): 275–314.

[10] DeMarzo, Peter M., Ilan Kremer, and Andrzej Skrzypacz. 2005. "Bidding with Securities: Auctions and Security Design." *American Economic Review* 95(4): 936–959.

[11] Fishman, Michael J. 1988. "A Theory of Pre-emptive Takeover Bidding." *RAND Journal of Economics* 19: 88–101.

[12] Gentry, Matthew and Caleb Stroup. 2019. "Entry and competition in takeover auctions." *Journal of Financial Economics* 132(2): 298–324.

[13] Gorbenko, Alexander S., and Andrey Malenko. 2011. "Competition among Sellers in Securities Auctions." *American Economic Review* 101(5): 1806–1841.

[14] Krasnokutskaya, Elena, and Katja Seim. 2011. "Bid preference programs and participation in highway procurement auctions." *American Economic Review* 101(6): 2653–2686.

[15] Krishna, Vijay. 2010. "Auction Theory." Second Edition, Elsevier: Academic Press.

[16] Levin, Dan, and James L. Smith. 1994. "Equilibrium in auctions with entry." *American Economic Review* 84(3): 585–599.

[17] Liu, Tingjun. 2016. "Optimal Equity Auctions with Heterogeneous Bidders." *Journal of Economic Theory* 166: 94–123.

[18] Lu, Jingfeng. 2009. "Auction design with opportunity cost." *Economic Theory* 38: 73–103.

[19] Marquez, Robert, and Rajdeep Singh. 2013. "The Economics of Club Bidding in Private Equity." *Journal of Financial Economics* 108(2): 493–505.

[20] McAfee, R. Preston, and John McMillan. 1987. "Auctions with entry." *Economics Letters* 23(4): 343–347.

[21] McAfee, R. Preston, and John McMillan. 1989. "Government procurement and international trade." *Journal of International Economics* 26(3–4): 291–308.

[22] Myerson, Roger B. 1981. "Optimal Auction Design." *Mathematics of Operations Research* 6(1): 58–73.

[23] Nakabayashi, Jun. 2013. "Small business set-asides in procurement auctions: An empirical analysis." *Journal of Public Economics* 100: 28–44.

[24] Povel, Paul and Rajdeep Singh. 2006. "Takeover Contest With Asymmetric Bidders." *Review of Financial Studies* 19(4): 1399–1431.

[25] Quint, Daniel and Kenneth Hendricks. 2018. "A theory of indicative bidding." *American Economic Journal: Microeconomics* 10(2): 118–151.

[26] Samuelson, W. 1985. "Competitive bidding with entry costs." *Economics Letters* 17 : 53–57.

[27] Skrzypacz, Andrzej. 2013. "Auctions with contingent payments: An overview." *International Journal of Industrial Organization* 31(5): 666–675.

[28] Sogo, Takeharu, Dan Bernhardt, and Tingjun Liu. 2016. "Endogenous Entry to Security-bid Auctions." *American Economic Review* 106(11): 3577–3589.

[29] Stegeman, Mark. 1996. "Participation costs and efficient auctions." *Journal of Economic Theory* 71(1): 228–259.

[30] Sweeting, Andrew and Vivek Bhattacharya. 2015. "Selective entry and auction design." *International Journal of Industrial Organization* 43: 189–207.

[31] Ye, Lixin. 2007. "Indicative bidding and a theory of two-stage auctions." *Games and Economic Behavior* 58(1): 181–207.